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Numerical Stability and Oscillation of a kind of Functional Differential Equations

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Abstract　The paper focuses on the stability and oscillation of numerical solutions for a kind of functional differential equations. Firstly, the conditions of numerical stability and oscillation are obtained by using the θ-methods. Secondly, we studied the preservation behavior of numerical methods for the two dynamical properties, namely under which conditions the stability and oscillation of the analytic solution can be inherited by numerical methods. Finally, some numerical examples are given.

Key words　Functional differential equations; θ-methods; numerical solution; stability; oscillation

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0　Introduction

As a special kind of functional differential equations, the differential equation with piecewise continuous arguments (DEPCA) has aroused lots of attention. Various properties of DEPCA have been investigated deeply. Such as convergence[16], stability [4,11], oscillation[2], periodicity[1], bifurcation[3] and asymptotic behavior[5], etc. However, all papers mentioned above deal with the properties of analytic solution of DEPCA. Nowadays, it is worth noting that numerical analysis of DEPCA be of particular interest for many scientists. Some important properties such as numerical stability[8], numerical oscillation[6,7] and numerical dissipativity[15] were investigated. In recent two papers [9,17], the authors discussed numerical approximation of DEPCA in stochastic and impulsive case, respectively. In the case of PDE, some our contributions[12,13] maybe noted. Different from above cases, in the present work, we shall study both numerical stability and oscillation for a more complicated DEPCA with scalar coefficients, and get some new results.

In this paper we consider the following DEPCA

$$u'(t) = au(t) + bu([t]) + cu(2[\frac{t+1}{2}]) \text{ , } u(0) = u_0 \text{ ,} \tag{1}$$

here a, b, c are all real coefficients and u_0 is initial condition, $[\cdot]$ denotes the greatest integer function. In particularly, when $c=0$, the equation in Eq. (1) becomes $u'(t) = au(t) + bu([t])$, which is exactly the case of [8]. If $b=0$, the equation in Eq. (1) becomes $u'(t) = au(t) + cu(2[(t+1)/2])$, which is exactly the case of [10]. Thus, the results in this paper are the generalization of corresponding ones in [8] and [10]. The following results for the analytical stability and oscillation of Eq. (1) will be useful for the upcoming analysis.

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Theorem 1[14] The analytic solution of Eq. (1) is asymptotically stable for any initial value, if one of the following cases is true

$$\begin{cases}(a+b+c)(e^a-1)((a+b-c)e^a-(b-a-c))<0,\\(a+b+c)(e^a-1)((a+b+c)e^a-(b-a+c))<0,\end{cases} c\neq a/(e^a-1),\quad a\neq 0 \tag{2}$$

and

$$(b+c)(b+2)(b^2+bc+2b+2)<0,\ c\neq 1,\quad a=0. \tag{3}$$

Theorem 2[14] Assume that $a\neq 0$, then Eq. (1) is oscillatory if and only if one of the following conditions is true

$$\begin{cases}b>-\dfrac{ae^a}{e^a-1}, c>\dfrac{a}{e^a-1},\\ b<-\dfrac{ae^a}{e^a-1}, c<\dfrac{a}{e^a-1}, b+c>-\dfrac{ae^a}{e^a-1},\\ b+c<-\dfrac{ae^a}{e^a-1}.\end{cases} \tag{4}$$

1 The discrete scheme

Set $h=1/m$ ($m\geqslant 1$) be stepsize, we consider the linear θ-method to Eq. (1)

$$u_{n+1}=u_n+h(\theta(au_{n+1}+bu^h([(n+1)h])+cu^h(2[\frac{(n+1)}{2}])) + (1-\theta)(au_n+bu^h([nh])+cu^h(2[\frac{nh+1}{2}]))) \tag{5}$$

and the one-leg θ-method to Eq. (1)

$$u_{n+1}=u_n+h(a(\theta u_{n+1}+(1-\theta)u_n)+bu^h([(n+\theta)h])+cu^h(2[\frac{(n+\theta)h+1}{2}])), \tag{6}$$

here $\theta\in[0,1]$, u_n, $u^h([nh])$ and $u^h(2[(nh+1)/2])$ denote approximations to $u(t)$, $u([t])$ and $u(2[(t+1)/2])$ at t_n, respectively.

Let $n=km+l(l=0,1,\cdots,m-1)$, we define $u^h(t_n+\eta h)$ as u_{km}, $u^h(2[(t_n+\eta h+1)/2])$ as u_{2km} according to [8,10], where $0\leqslant\eta\leqslant 1$. Thus Eqs. (5) and (6) can be reduced to the same recurrence relation

$$u_{km+l+1}=\begin{cases}\alpha u_{km+l}+\gamma_1 u_{km},\ k \text{ is even},\\ \alpha u_{km+l}+\beta u_{km}+\gamma_2 u_{(k+1)m},\ k \text{ is odd},\end{cases} \tag{7}$$

where $\alpha=1+\dfrac{ha}{1-\theta ha}$, $\beta=\dfrac{hb}{1-\theta ha}$, $\gamma_1=\dfrac{h(b+c)}{1-\theta ha}$, $\gamma_2=\dfrac{hc}{1-\theta ha}$.

It is easily seen that Eq. (7) is equivalent to the following two cases

(1) If $a\neq 0$,

$$u_{km+l+1}=\begin{cases}(\alpha^{l+1}+\dfrac{b+c}{a}(\alpha^{l+1}-1))u_{km},\ k \text{ is even},\\ (\alpha^{l+1}+\dfrac{b}{a}(\alpha^{l+1}-1))u_{km}+\dfrac{c}{a}(\alpha^{l+1}-1)u_{(k+1)m},\ k \text{ is odd},\end{cases} \tag{8}$$

$$u_n=\begin{cases}(\alpha^l+\dfrac{b+c}{a}(\alpha^l-1))u_{km},\ k \text{ is even},\\ (\alpha^l+\dfrac{b}{a}(\alpha^l-1))u_{km}+\dfrac{c}{a}(\alpha^l-1)u_{(k+1)m},\ k \text{ is odd}.\end{cases} \tag{9}$$

(2) If $a=0$,

$$u_{km+l+1}=\begin{cases}(1+h(l+1)(b+c))u_{km},\ k \text{ is even},\\ (1+h(l+1)b)u_{km}+h(l+1)cu_{(k+1)m},\ k \text{ is odd},\end{cases} \tag{10}$$

$$u_n=\begin{cases}(1+hl(b+c))u_{km},\ k \text{ is even},\\ (1+hlb)u_{km}+hlcu_{(k+1)m}, k \text{ is odd}.\end{cases} \tag{11}$$

Theorem 3 Assume that $\lambda\neq 0$, then Eq. (1) has the numerical solution

$$u_n=\begin{cases}((\alpha^m+\dfrac{b+c}{a}(\alpha^m-1))\alpha^l+\dfrac{1}{a}(\alpha^l-1)(b\alpha^m+\dfrac{b+c}{a}(b\alpha^m-b)+\lambda c))\lambda^{j-1}u_0, n=(2j-1)m+l,\ j=1,2,\cdots,\\ (\alpha^l+\dfrac{b+c}{a}(\alpha^l-1))\lambda^j u_0, n=2jm+l, j=0,1,2,\cdots\end{cases} \tag{12}$$

for $a \neq 0$, where $l = 0,1,2,\cdots,m-1$ and

$$\lambda = \frac{(\alpha^m + b(\alpha^m - 1)/a)(\alpha^m + (b+c)(\alpha^m - 1)/a)}{1 - c(\alpha^m - 1)/a}$$

and

$$u_n = \begin{cases} ((b^2 + bc + b + \lambda c)hl + b + c + 1)\lambda^{j-1}u_0, \ n = (2j-1)m + l, \ j = 1,2,\cdots, \\ ((b+c)hl + 1)\lambda^j u_0, \ n = 2jm + l, \ j = 0,1,2,\cdots \end{cases} \tag{13}$$

for $a = 0$, where $l = 0,1,2,\cdots,m-1$ and $\lambda = (b+1)(b+c+1)/(1-c)$.

Proof Assume that u_n is a solution of Eq. (7) with conditions $u_{2jm} = d_{2j}$ and $u_{(2j-1)m} = d_{2j-1}$. It follows from (9) that

$$u_n = \begin{cases} (\alpha^l + \frac{b}{a}(\alpha^l - 1))u_{(2j-1)m} + \frac{c}{a}(\alpha^l - 1)u_{2jm}, \ n = (2j-1)m + 1, \ j = 1,2,\cdots, \\ (\alpha^l + \frac{b+c}{a}(\alpha^l - 1))u_{2jm}, \ n = 2jm + l, \ j = 0,1,2,\cdots. \end{cases} \tag{14}$$

From (14) and (8) with $l = m-1$ we have

$$(1 - \frac{c}{a}(\alpha^m - 1))u_{2jm} = (\alpha^m + \frac{b}{a}(\alpha^m - 1))u_{(2j-1)m}, j = 1,2,\cdots,$$

$$u_{(2j+1)m} = (\alpha^m + \frac{b+c}{a}(\alpha^m - 1))u_{2jm}, j = 0,1,2,\cdots,$$

which implies that

$$\begin{aligned} d_{2j} &= \frac{\alpha^m + b(\alpha^m - 1)/a}{1 - c(\alpha^m - 1)/a} d_{2j-1}, j = 1,2,\cdots, \\ d_{2j+1} &= (\alpha^m + \frac{b+c}{a}(\alpha^m - 1))d_{2j}, j = 0,1,2,\cdots, \end{aligned} \tag{15}$$

hence

$$d_{2j+1} = \frac{(\alpha^m + b(\alpha^m - 1)/a)(\alpha^m + (b+c)(\alpha^m - 1)/a)}{1 - c(\alpha^m - 1)/a} d_{2j-1} = \lambda d_{2j-1},$$

then if $a \neq 0$ and $\lambda \neq 0$ we get (12). Formula (13) can be obtained in the same way. The proof is complete.

2 Stability and oscillation of numerical solution

Lemma 1 $u_n \to 0$ as $n \to \infty$ if and only if $|\lambda| < 1$, where λ is defined in Theorem 3.

Theorem 4 The numerical solution of Eq. (1) is asymptotically stable for u_0 , if one of the following cases is true

$$\begin{cases} (a+b+c)(\alpha^m - 1)((a+b-c)\alpha^m - (b-a-c)) < 0, \\ (a+b+c)(\alpha^m - 1)((a+b+c)\alpha^m - (b-a+c)) < 0, \end{cases} c \neq a/(\alpha^m - 1), a \neq 0 \tag{16}$$

and

$$(b+c)(b+2)(b^2 + bc + 2b + 2) < 0, c \neq 1, \quad a = 0. \tag{17}$$

Proof According to Lemma 1 and Theorem 3 we know that the numerical solution of Eq. (1) is asymptotically stable if and only if

$$\left| \frac{(\alpha^m + b(\alpha^m - 1)/a)(\alpha^m + (b+c)(\alpha^m - 1)/a)}{1 - c(\alpha^m - 1)/a} \right| < 1 \tag{18}$$

for $a \neq 0$ and

$$\left| \frac{(b+1)(b+c+1)}{1-c} \right| < 1 \tag{19}$$

for $a = 0$.

If $a \neq 0$, from (18) we have the following two inequalities hold

$$\left| \frac{\alpha^m + b(\alpha^m - 1)/a}{1 - c(\alpha^m - 1)/a} \right| < 1, \quad \left| \alpha^m + \frac{b+c}{a}(\alpha^m - 1) \right| < 1,$$

which are equivalent to

$$\begin{cases}\left(\dfrac{\alpha^m+\dfrac{b}{a}(\alpha^m-1)}{1-\dfrac{c}{a}(\alpha^m-1)}+1\right)\left(\dfrac{\alpha^m+\dfrac{b}{a}(\alpha^m-1)}{1-\dfrac{c}{a}(\alpha^m-1)}-1\right)<0,\\ \left(\alpha^m+\dfrac{b+c}{a}(\alpha^m-1)+1\right)\left(\alpha^m+\dfrac{b+c}{a}(\alpha^m-1)-1\right)<0,\end{cases}$$

so we can get (16). If $a=0$, similar to the case of $a\neq 0$, we can get (17) from (19).

From Theorem 3 we have the following corollary.

Corollary 1　Assume that $a\neq 0$ then

$$u_n=\begin{cases}\left(\alpha^l+\dfrac{b}{a}(\alpha^l-1)\right)d_{2j-1}+\dfrac{c}{a}(\alpha^l-1)d_{2j},n=(2j-1)m+l,j=1,2,\cdots,\\ \left(\alpha^l+\dfrac{b+c}{a}(\alpha^l-1)\right)d_{2j},n=2jm+l,j=0,1,2,\cdots,\end{cases}$$

where $d_j=u_{jm}$ and satisfies (15).

It is easy to check that the following two lemmas are hold.

Lemma 2　Sequence $u_l=-a\alpha^l/(\alpha^l-1)$ is strictly monotonic increasing for $l=0,1,\cdots,m$ and $a\neq 0$.

Lemma 3　Assume that $b+c>-a\alpha^m/(\alpha^m-1)$ holds, then $b+c>-a\alpha^l/(\alpha^l-1)$ holds for $l=0,1,\cdots,m-1$ implies that $\alpha^l+(b+c)(\alpha^l-1)/a>0$.

Theorem 5　Assume that $a\neq 0$, then Eq. (7) is oscillatory if and only if any of the following conditions is satisfied

$$\begin{cases}b>-\dfrac{a\alpha^m}{\alpha^m-1},\ c>\dfrac{a}{\alpha^m-1},\\ b<-\dfrac{a\alpha^m}{\alpha^m-1},\ c<\dfrac{a}{\alpha^m-1},\ b+c>-\dfrac{a\alpha^m}{\alpha^m-1},\\ b+c<-\dfrac{a\alpha^m}{\alpha^m-1}.\end{cases}\tag{20}$$

Proof　Sufficiency. It follows from (19) that the sequence $\{d_j\}$ oscillates under any of the condition (20). Since $u_{jm}=d_j$ for $j=0,1,\cdots$, so u_n also oscillates.

Necessity. We assume that any of the following hypotheses is satisfied

$$\begin{cases}b<-\dfrac{a\alpha^m}{\alpha^m-1},\ c>\dfrac{a}{\alpha^m-1},\ b+c>-\dfrac{a\alpha^m}{\alpha^m-1},\\ b>-\dfrac{a\alpha^m}{\alpha^m-1},\ c<\dfrac{a}{\alpha^m-1},\ b+c>-\dfrac{a\alpha^m}{\alpha^m-1}.\end{cases}\tag{21}$$

Let u_n be the solution of Eq. (7), then from (15) and (21) we have $d_j>0$ for $j=0,1,2,\cdots$. By Corollary 1, Lemmas 2 and 3 we get

(1) For $n=2jm+l$ and $j=0,1,2,\cdots$

$$u_n=\left(\alpha^l+\frac{b+c}{a}(\alpha^l-1)\right)d_{2j}>0.$$

(2) For $n=(2j-1)m+l$ and $j=1,2,\cdots$

$$u_n=\alpha^l d_{2j-1}+\frac{1}{a}(\alpha^l-1)(bd_{2j-1}+cd_{2j})=\left(d_{2j-1}+\frac{b}{a}d_{2j-1}+\frac{c}{a}d_{2j}\right)\alpha^l-\frac{b}{a}d_{2j-1}-\frac{c}{a}d_{2j}.$$

So we know that u_n is a monotonous sequence for $n=(2j-1)m+l$. On the other hand, $u_{(2j-1)m+m}=d_{2j}$ at $l=m$, so u_n has the minimum value d_{2j} or d_{2j-1} for $n=(2j-1)m+l$, namely

$$u_n\geqslant\min\{d_{2j-1},d_{2j}\}>0.$$

Combining (i) with (ii), we obtain $u_n>0$ for $n=km+l$. This contradicts the assumption that u_n oscillates. The proof is complete.

3 Preservation of stability

***Definition* 1** The set of all triples (a,b,c) which satisfy the condition (2) is called an asymptotical stability region denoted by H.

Followed by Definition 1, the numerical asymptotical stability region can be denoted by S.

Lemma 4[18] Let $\varphi(x)=1/x-1/(e^x-1)$, then $\varphi(x)$ is a decreasing function and $\varphi(-\infty)=1$, $\varphi(0)=1/2$ and $\varphi(+\infty)=0$.

Lemma 5[18] For all $m>|a|$,

$(1+a/(m-\theta a))^m\geqslant e^a$ if and only if $1/2\leqslant\theta\leqslant 1$ for $a>0$, $\varphi(-1)\leqslant\theta\leqslant 1$ for $a<0$;

$(1+a/(m-\theta a))^m\leqslant e^a$ if and only if $0\leqslant\theta\leqslant 1/2$ for $a<0$, $0\leqslant\theta\leqslant\varphi(1)$ for $a>0$,

where $\varphi(x)=1/x-1/(e^x-1)$.

Lemma 6 For all $m>M$,

(1) $(1+a/(m-\theta a))^m\geqslant e^a$ if and only if $1/2\leqslant\theta\leqslant 1$ for $a>0$, $\varphi(a/M)\leqslant\theta\leqslant 1$ for $a<0$;

(2) $(1+a/(m-\theta a))^m\leqslant e^a$ if and only if $0\leqslant\theta\leqslant 1/2$ for $a<0$, $0\leqslant\theta\leqslant\varphi(a/M)$ for $a>0$,

where $\varphi(x)=1/x-1/(e^x-1)$.

Proof (i) From $(1+a/(m-\theta a))^m\geqslant e^a$ we have $\theta\geqslant m/a-1/(e^{a/m}-1)$. So for all $m>M$, in view of Lemma 4 we obtain $1/2\leqslant\theta\leqslant 1$ for $a>0$, $\varphi(a/M)\leqslant\theta\leqslant 1$ for $a<0$. The case of (ii) can be proved in the same way.

Lemma 7 Assume that inequality $p<e^a<q$ holds for all $a\neq 0$, then inequality $p<\alpha^m<q$ also holds if any of the following conditions is satisfied:

(1) $1/2\leqslant\theta\leqslant 1$ or $0\leqslant\theta\leqslant\varphi(1)$ for $m\geqslant M$ and $a>0$;

(2) $\varphi(-1)\leqslant\theta\leqslant 1$ or $0\leqslant\theta\leqslant 1/2$ for $m\geqslant M$ and $a<0$,

where $\varphi(x)=1/x-1/(e^x-1)$.

Proof For all $a\neq 0$, there exists a $M_0>0$, when $m>M_0$, the range of α^m has the following two cases

$$e^a\leqslant\alpha^m<q \text{ and } p<\alpha^m\leqslant e^a.$$

Let $\varepsilon=\min\{q-e^a,e^a-p\}$, if $e^a\leqslant\alpha^m<q$, there is

$$M_1^*=\inf\{m:m\ln(1+1/(m/a-\theta))<\ln(e^a+\varepsilon)\}+1,$$

such that $\alpha^m-e^m<\varepsilon$, which implies that $\alpha^m<q$ whenever $m\geqslant M_1^*$. Let $M_1=\max\{|a|,M_1^*\}$, then for all $m\geqslant M_1$, from Lemma 6 we obtain that the Inequality $e^a\leqslant\alpha^m<q$ holds under the conditions $1/2\leqslant\theta\leqslant 1$ for $a>0$ and $\varphi(-1)\leqslant\theta\leqslant 1$ for $a<0$.

If $p<\alpha^m\leqslant e^a$, there is $M_2^*=\inf\{m:(1+1/(m/a-\theta))^m>e^a-\varepsilon\}+1$ such that $\alpha^m-e^a>-\varepsilon$, which implies that $\alpha^m>p$ whenever $m\geqslant M_2^*$.

Let $M_2=\max\{|a|,M_2^*\}$, then for all $m\geqslant M_2$, from Lemma 6 we have that the Inequality $p<\alpha^m\leqslant e^a$ holds under the conditions $0\leqslant\theta\leqslant 1/2$ for $a<0$ and $0\leqslant\theta\leqslant\varphi(1)$ for $a>0$. Set $M=\max\{M_1,M_2\}$, then the proof is complete.

We will investigate which condition leads to $H\subseteq S$. For convenience, we divide the region H into five parts

$H_0=\{(0,b,c)\in H:a=0\}$, $H_1=\{(a,b,c)\in H\backslash H_0:a>0,(a+b+c)(a+b-c)>0\}$,

$H_2=\{(a,b,c)\in H\backslash H_0:a>0,(a+b+c)(a+b-c)<0\}$, $H_3=\left\{\begin{matrix}(a,b,c)\in H\backslash H_0:a<0,\\(a+b+c)(a+b-c)>0\end{matrix}\right\}$,

$H_4=\{(a,b,c)\in H\backslash H_0:a<0,(a+b+c)(a+b-c)<0\}$.

In the similar way, we denote

$S_0=\{(0,b,c)\in S:a=0\}$, $S_1=\{(a,b,c)\in S\backslash S_0:a>0,(a+b+c)(a+b-c)>0\}$,

$S_2=\{(a,b,c)\in S\backslash S_0:a>0,(a+b+c)(a+b-c)<0\}$, $S_3=\left\{\begin{matrix}(a,b,c)\in S\backslash S_0:a<0,\\(a+b+c)(a+b-c)>0\end{matrix}\right\}$,

$S_4=\{(a,b,c)\in S\backslash S_0:a<0,(a+b+c)(a+b-c)<0\}$.

It is easy to see that $H=\bigcup_{i=0}^{4}H_i$, $S=\bigcup_{i=0}^{4}S_i$ and

$$H_i\cap H_j=\varnothing,\ S_i\cap S_j=\varnothing,\ H_i\cap S_j=\varnothing,\ i\neq j,i,j=0,1,2,3,4.$$

Therefore, we can conclude that $H\subseteq S$ is equivalent to $H_i\subseteq S_i, i=0,1,2,3,4$.

If $a>0$, (2) becomes

$$\begin{cases}(a+b+c)((b-c-a)-(a+b+c)e^a)>0,\\(a+b+c)((b+c-a)-(a+b+c)e^a)>0.\end{cases}\tag{22}$$

(16) yields

$$\begin{cases}(a+b+c)((b-c-a)-(a+b+c)\alpha^m)>0,\\(a+b+c)((b+c-a)-(a+b+c)\alpha^m)>0.\end{cases}\tag{23}$$

If $a<0$, (2) turns into

$$\begin{cases}(a+b+c)((a+b-c)e^a-(b-c-a))>0,\\(a+b+c)((a+b+c)e^a-(b+c-a))>0.\end{cases}\tag{24}$$

(16) gives

$$\begin{cases}(a+b+c)((a+b-c)\alpha^m-(b-c-a))>0,\\(a+b+c)((a+b+c)\alpha^m-(b+c-a))>0.\end{cases}\tag{25}$$

Theorem 6 The stability regions have the following five relationships

$H_0\subseteq S_0$ if and only if $0\leqslant\theta\leqslant1$; $H_1\subseteq S_1$ if and only if $0\leqslant\theta\leqslant\varphi(1)$; $H_2\subseteq S_2$ if $1/2\leqslant\theta\leqslant1$ or $0\leqslant\theta\leqslant\varphi(1)$; $H_3\subseteq S_3$ if and only if $\varphi(-1)\leqslant\theta\leqslant1$; $H_4\subseteq S_4$ if $\varphi(-1)\leqslant\theta\leqslant1$ or $0\leqslant\theta\leqslant1/2$, where $\varphi(x)=1/x-1/(e^x-1)$.

Proof (i) Noticing that (3) and (17) are the same in form, so $H_0\subseteq S_0$ holds for all θ with $0\leqslant\theta\leqslant1$.

(ii) By the notation of H_1 and S_1, (22) yields

$$e^a<\frac{b-c-a}{b-c+a},\ e^a<\frac{b+c-a}{b+c+a}.$$

(23) can be changed into

$$\alpha^m<\frac{b-c-a}{b-c+a},\alpha^m<\frac{b+c-a}{b+c+a}.$$

Therefore, $H_1\subseteq S_1$ if and only if $\alpha^m\leqslant e^a$, so by Lemma 5 we have $H_1\subseteq S_1$ if and only if $0\leqslant\theta\leqslant\varphi(1)$.

(iii) By the notation of H_2 and S_2 , (22) becomes

$$\frac{b-c-a}{b-c+a}<e^a<\frac{b+c-a}{b+c+a}.$$

(23) gives

$$\frac{b-c-a}{b-c+a}<\alpha^m<\frac{b+c-a}{b+c+a}.$$

Let $p=\dfrac{b-c-a}{b-c+a}, q=\dfrac{b+c-a}{b+c+a}$, then by Lemma 7 we have $H_2\subseteq S_2$ if $1/2\leqslant\theta\leqslant1$ or $0\leqslant\theta\leqslant\varphi(1)$.

(iv) and (v) can be proved in the similar way.

4 Preservation of oscillation

Definition 2 We call the θ-methods preserve oscillation of Eq. (1) if Eq. (1) oscillates, which implies that there is an h_0 such that Eq. (7) oscillates for $h<h_0$.

By some easy inductions we have the next lemma.

Lemma 8 For all $m>|a|$ and $\theta\in[0,1]$,

(1) $-a-1<-a\alpha^m/(\alpha^m-1)<-a$ and $0<a/(\alpha^m-1)<1$ for $a>0$;

(2) $-1<-a\alpha^m/(\alpha^m-1)<0$ and $-a<a/(\alpha^m-1)<-a+1$ for $a<0$.

The following lemma can be naturally obtained from Theorem 2.

Lemma 9 Eq. (1) is oscillatory if any of the following conditions is satisfied

(1) $b+c<-ae^a/(e^a-1)$, $b\geqslant-a$ and $c>a/(e^a-1)$ for $a>0$;

(2) $b+c<-ae^a/(e^a-1)$, $b\geqslant0$ and $c>a/(e^a-1)$ for $-\ln2\leqslant a<0$;

(3) $b+c<-ae^a/(e^a-1)$, $b\geqslant 0$ and $c>a/(e^a-1)$, $b+c>0$, $b<-ae^a/(e^a-1)$ and $c\leqslant -a$ for $a<-\ln 2$.

By Theorem 5, Lemmas 8 and 9 the following corollary is obtained.

Corollary 2 For all $m>|a|$, under the condition of Lemma 9, if

$$-\frac{ae^a}{e^a-1}\leqslant-\frac{a\alpha^m}{\alpha^m-1}\text{ or }\frac{a}{e^a-1}\geqslant\frac{a}{\alpha^m-1}$$

holds, then the numerical solutions inherit oscillation of the analytic solutions of Eq. (1).

So we have the first result for preservation of oscillation.

Theorem 7 The numerical solutions inherit oscillation of the analytic solutions of Eq. (1) if one of the following conditions holds

(1) $1/2\leqslant\theta\leqslant 1$ for $a>0$;

(2) $0\leqslant\theta\leqslant 1/2$ for $a<0$.

Proof From Corollary 2 we obtain that the numerical solutions inherit oscillation of the analytic solutions of Eq. (1) if $\alpha^m\geqslant e^a$ for $a>0$ and $\alpha^m\leqslant e^a$ for $a<0$. Then by Lemma 5 the proof is finished.

Furthermore, from Theorem 2 we can easily obtain the following lemma.

Lemma 10 Eq. (1) is oscillatory if any of the following conditions is satisfied

(1) $b+c<-a-1$, $b>-ae^a/(e^a-1)$ and $c\geqslant 1$ for $a>a_1$;

(2) $b+c<-a-1$, $b>-ae^a/(e^a-1)$ and $c\geqslant 1$, $b+c>-ae^a/(e^a-1)$, $b\leqslant -a-1$, $c<a/(e^a-1)$ for $0<a\leqslant a_1$;

(3) $b+c\leqslant -1$, $b>-ae^a/(e^a-1)$ and $c\geqslant -a+1$, $b+c>-ae^a/(e^a-1)$, $b\leqslant -1$, $c<a/(e^a-1)$ for $a<0$,

where a_1 is the positive root of equation $e^a-2a-1=0$.

By Theorem 5, Lemmas 8 and 10 the following corollary is got.

Corollary 3 For all $m>|a|$, under the condition of Lemma 10, if

$$-\frac{ae^a}{e^a-1}\geqslant-\frac{a\alpha^m}{\alpha^m-1}\text{ or }\frac{a}{e^a-1}\leqslant\frac{a}{\alpha^m-1}$$

holds, then the numerical solutions inherit oscillation of the analytic solutions of Eq. (1).

So we have the second result for preservation of oscillation.

Theorem 8 The numerical solutions inherit oscillation of the analytic solutions of Eq. (1) if one of the following conditions holds (1) $0\leqslant\theta\leqslant\varphi(1)$ for $a>0$; (2) $\varphi(-1)\leqslant\theta\leqslant 1$ for $a<0$, where $\varphi(x)=1/x-1/(e^x-1)$.

Proof From Corollary 3 we obtain that the numerical solutions inherit oscillation of the analytic solutions of Eq. (1) if $\alpha^m\leqslant e^a$ for $a>0$ and $\alpha^m\geqslant e^a$ for $a<0$. Then by Lemma 5 the proof is completed.

Since $\lim_{m\to\infty}\alpha^m=e^a$ so

$$\lim_{m\to\infty}\frac{a}{\alpha^m-1}=\frac{a}{e^a-1}\text{ , }\lim_{m\to\infty}\left(-\frac{a\alpha^m}{\alpha^m-1}\right)=-\frac{ae^a}{e^a-1}\text{ .}$$

Let $\varepsilon=\min\left\{\left|b+\frac{ae^a}{e^a-1}\right|,\left|c-\frac{a}{e^a-1}\right|,\left|\frac{1}{\sqrt{2}}\left(b+c+\frac{ae^a}{e^a-1}\right)\right|\right\}$, $M^*=\inf\left\{m:\left|\frac{\alpha^m-e^a}{\alpha^m-1}\right|<\frac{\varepsilon(e^a-1)}{a}\right\}+1$ and $M=\max\{|a|,M^*\}$ such that

$$\left|\frac{a}{\alpha^m-1}-\frac{a}{e^a-1}\right|<\varepsilon\text{ ,}\quad\left|-\frac{a\alpha^m}{\alpha^m-1}+\frac{ae^a}{e^a-1}\right|<\varepsilon\text{ ,}$$

so the third result for preservation of oscillation is as follows.

Theorem 9 The numerical solutions inherit oscillation of the analytic solutions of Eq. (1) if one of the following conditions holds (1) $0\leqslant\theta\leqslant\varphi(1)$ or $1/2\leqslant\theta\leqslant 1$ for $a>0$ and $m\geqslant M$; (2) $\varphi(-1)\leqslant\theta\leqslant 1$ or $0\leqslant\theta\leqslant 1/2$ for $a<0$ and $m\geqslant M$, where $\varphi(x)=1/x-1/(e^x-1)$.

Proof It is easy to know that the range of $a/(\alpha^m-1)$ has two cases for all $m\geqslant M$, $a/(\alpha^m-1)\leqslant a/(e^a-1)$ and (ii) $a/(\alpha^m-1)\geqslant a/(e^a-1)$.

The first case implies that $\alpha^m \geqslant e^a$ for $a > 0$ and $\alpha^m \leqslant e^a$ for $a < 0$, then by Lemmas 5, 6 we have that the numerical solutions inherit oscillation of Eq. (1) if $1/2 \leqslant \theta \leqslant 1$ for $a > 0$ and $0 \leqslant \theta \leqslant 1/2$ for $a < 0$.

The second case can be obtained similarly. The proof is complete.

Remark 1 It is easy to see that $u_n = u(t_n)$ for $a = 0$, so the preservation of oscillation of the θ-methods is obvious when $a = 0$.

5 Numerical simulations

We propose some examples to test the above main results.

Consider the equation

$$u'(t) = u(t) + u([t]) - 3u(2\left[\frac{t+1}{2}\right]),\quad u(0) = 1. \tag{26}$$

Let $m = 100$ and $\theta = 0.5$, it is can be seen that the condition (16) is satisfied for $a = 1, b = 1, c = -3$. In Figure 1, we draw the figure of the numerical solution of Eq. (26), from this figure we know that the numerical solution of Eq. (26) is asymptotically stable, which is in agreement with Theorem 4.

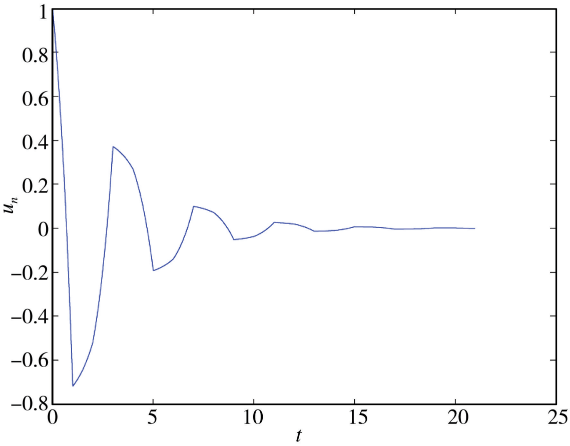

Figure 1 The numerical solutions of Eq. (26) with $\theta = 0.5$ and $m = 100$

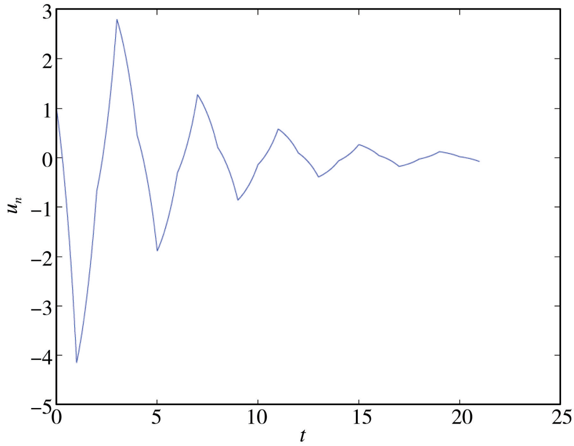

Figure 2 The numerical solutions of Eq. (27) with $\theta = 0.4$ and $m = 100$

For the equation

$$u'(t) = u(t) - u([t]) - 3u(2\left[\frac{t+1}{2}\right]),\quad u(0) = 1. \tag{27}$$

We can test that $a = 1, b = -1, c = -3$, $m = 100$ and $\theta = 0.4$ satisfy the third condition in (20). In Figure 2, we draw the figure of the numerical solution of Eq. (27), we can see that the numerical solution of Eq. (27) is oscillatory, which coincides with Theorem 5.

We consider the equation

$$u'(t) = u(t) + 1.5u([t]) - 3u(2\left[\frac{t+1}{2}\right]),\quad u(0) = 1, \tag{28}$$

it is easy to see that $a = 1, b = 1.5, c = -3$, $m = 100 > M = 2$ and $\theta = 0.4$ satisfy (iii) in Theorem 6. We gave the analytic solutions and the numerical solutions of Eq. (28) in Figure 3, we can easily see that the numerical solutions inherit the stability of analytic solutions of Eq. (28), which in accordance with Theorem 6.

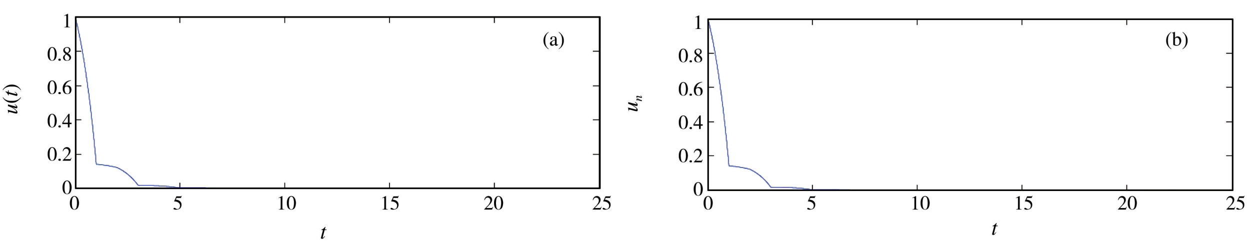

Figure 3 The analytic solutions (a) and the numerical solutions (b) of Eq. (28) with $\theta = 0.4$ and $m = 100$

Consider the equation

$$u'(t) = -u(t) - u([t]) + u(2\left[\frac{t+1}{2}\right]),\quad u(0) = 1. \tag{29}$$

Let $m = 100$ and $\theta = 0.7$, it can be checked that $a = -1, b = -1, c = 1$ satisfy (iv) in Theorem 6. In Figure 4, we draw the figures of the analytic solution and the numerical solution of Eq. (29), respectively, we can see that the θ-methods preserve the stability of Eq. (29), which is in agreement with Theorem 6.

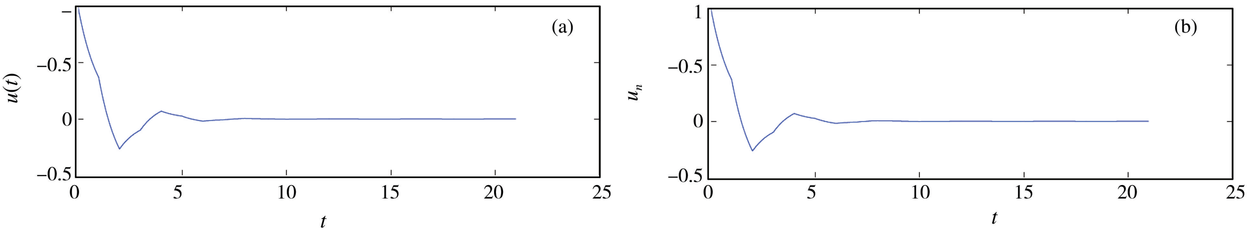

Figure 4　The analytic solutions (a) and the numerical solutions (b) of Eq. (29) with $\theta = 0.7$ and $m = 100$

For the equation

$$u'(t) = -u(t) - 0.8u([t]) + u(2\left[\frac{t+1}{2}\right]),\quad u(0) = 1, \tag{30}$$

it is not difficult to see that $a = -1, b = -0.8, c = 1$ satisfy (iii) in Lemma 9. Let $m = 100$ and $\theta = 0.4$. We gave the analytic solutions and the numerical solutions of Eq. (30) in Figure 5. It shows that the numerical solutions inherit the oscillation of analytic solutions of Eq. (30), which coincides with Theorem 7.

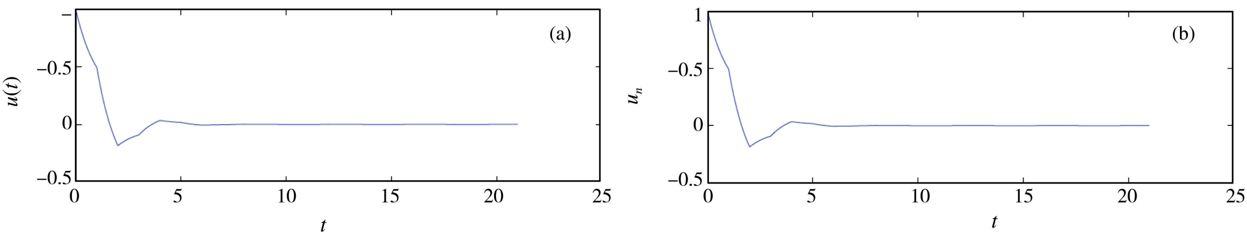

Figure 5　The analytic solutions (a) and the numerical solutions (b) of Eq. (30) with $\theta = 0.4$ and $m = 100$

Furthermore, for the equation

$$u'(t) = u(t) + u([t]) - 5u(2\left[\frac{t+1}{2}\right]), u(0) = 1. \tag{31}$$

We can verify that the coefficients $a = 1, b = 1, c = -5$ satisfy (iii) in Lemma 10. Let $m = 100 > M = 2$ and $\theta = 0.8$, in Figure 6, we draw the figures of the analytic solution and the numerical solution of Eq. (31), respectively, we can see that the θ-methods preserve the oscillation of Eq. (31), which is in agreement with Theorem 9.

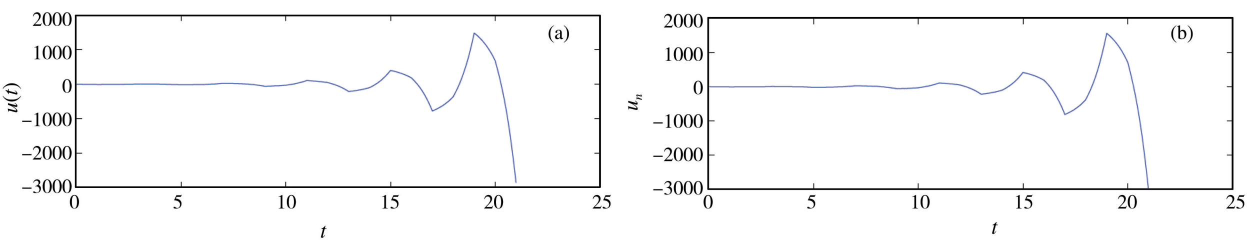

Figure 6　The analytic solutions (a) and the numerical solutions (b) of Eq. (31) with $\theta = 0.8$ and $m = 100$

The other results can be tested in the same way. All numerical results show good agreement with our theoretical results.

6　Conclusions

In this paper, we consider the numerical properties of θ-methods for a special kind of functional differential equations. Some conditions for the stability and oscillation of the numerical solution are given. The conditions that the θ-methods preserve the stability and oscillations of the analytic solutions are obtained. The numerical examples show that the θ-methods are suitable and effective for solving this kind of equation. We will consider the multidimensional case and the linear multistep methods in our further work.

References

[1] Akhmet M U,Fen M O,Kirane M. Almost periodic solutions of retarded SICNNs with functional response on piecewise constant argument[J]. Neural Comput Appl,2016,27:2483-2495.

[2] Bereketoglu H,Seyhan G,Ogun A. Advanced impulsive differential equations with piecewise constant arguments[J]. Math Model Anal,2010:15,175-187.

[3] Cavalli F,Naimzada A. A multiscale time model with piecewise constant argument for a boundedly rational monopolist[J]. J Differ Equ Appl,2016,22:1480-1489.

[4] Chiu K S. Exponential stability and periodic solutions of impulsive neural network models with piecewise constant argument[J]. Acta Appl Math,2017,151:199-226 .

[5] Karakoc F. Asymptotic behaviour of a population model with piecewise constant argument[J]. Appl Math Lett,2017,70:7-13.

[6] Liu M Z,Gao J F,Yang Z W. Oscillation analysis of numerical solution in the θ-methods for equation $x'(t)+ax(t)+a_1x([t-1])=0$[J]. Appl Math Comput,2007,186,566-578.

[7] Liu M Z,Gao J F,Yang Z W. Preservation of oscillations of the Runge-Kutta method for equation $x'(t)+ax(t)+a_1x([t-1])=0$[J]. Comput Math Appl,2009,58:1113-1125.

[8] Liu M Z,Song M H,Yang Z W. Stability of Runge-Kutta methods in the numerical solution of equation $u'(t)=au(t)+a_0u([t])$[J]. J Comput Appl Math,2005,166:361-370.

[9] Milosevic M. The Euler-Maruyama approximation of solutions to stochastic differential equations with piecewise constant arguments[J]. J Comput Appl Math,2016,298:1-12.

[10] Song M H,Liu M Z. Numerical stability and oscillation of the Runge-Kutta methods for the differential equations with piecewise continuous arguments alternately of retarded and advanced type[J]. J Inequal Appl,2012,290:2012.

[11] Wan L G,Wu A L. Multistability in Mittag-Leffler sense of fractional-order neural networks with piecewise constant arguments [J]. Neurocomputing,2018,286:1-10.

[12] Wang Q. Stability analysis of parabolic partial differential equations with piecewise continuous arguments[J]. Numer Meth Part D E,2017,33:531-545.

[13] Wang Q,Wang X M. Stability of θ-schemes for partial differential equations with piecewise constant arguments of alternately retarded and advanced type[J]. Int J Comput Math,2019,96:2352-2370.

[13] Wang Q,Wang X M. Stability and oscillation of mixed differential equation with piecewise continuous arguments[J]. Asia Pac J Math,2018,5:50-59.

[14] Wang W S,Li S F. Dissipativity of Runge-Kutta methods for neutral delay differential equations with piecewise constant delay[J]. Appl Math Lett,2008,21:983-991.

[15] Wu A L,Zeng Z G. Output convergence of fuzzy neurodynamic system with piecewise constant argument of generalized type and time-varying input[J]. IEEE T Syst Man Cy-S,2016,46:1689-1702.

[16] Zhang G L. Stability of Runge-Kutta methods for linear impulsive delay differential equations with piecewise constant arguments [J]. J Comput Appl Math,2016,297:41-50.

[17] Song M H,Yang Z W,Liu M Z. Stability of θ-methods for advanced differential equations with piecewise continuous arguments[J]. Comput Math Appl,2005,49:1295-1301.

一类泛函微分方程的数值稳定性和振动性

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摘　要　考虑一类泛函微分方程数值解的稳定性和振动性. 首先,用θ-方法求解方程,获得了数值解稳定和振动的条件. 接下来研究了数值方法对上述两种动力学行为的保持性质,得到了解析解的稳定性和振动性被数值方法保持的条件. 最后给出一些数值算例.

关键词　泛函微分方程;θ-方法;数值解;稳定性;振动性