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Oscillatory Citerion for Nonlinear Conformable Fractional Differential Equations

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Abstract In this paper, we give an oscillation criterion of interval type for second order nonlinear conformable fractional differential equations. This oscillation criterion relies on information of the coefficients on a subsequence of the positive half-line, rather than the whole positive half-line.

Key words Conformable fractional differential equation; oscillation criterion; Riccati transformation

0 Introduction

Fractional differential equations are used in various fields such as physics, mathematics, biology, biomedical sciences, finance as well as other disciplines (see [2,7]). In the last few decades, many people have been studying fractional differential equations. There are many definitions in the existent literature, such as the Riemann-Liouville, Caputo, Riesz, Riesz-Caputo, Weyl, Grunwald-Letnikov, Hadamard, and Chen derivatives. In 2004, R. Khalil et al. (see [4]) have suggested a new fractional derivative, which is called conformable derivative. This new definition has a lot of very nice properties, many properties that are consistent with the standard derivative, such as the definition itself, the chain rule that we are familiar with. In this paper, we shall investigate the oscillation criterion for the second order nonlinear conformable fractional differential equation

$$(p(t)y^{(\alpha)}(t))^{(\alpha)} + q(t)f(y(t)) = e(t),\ t \geqslant t_0 > 0, \tag{1}$$

where $p \in C^1([t_0,\infty),(0,\infty))$, $q,e \in C([t_0,\infty),R)$, $0 < \alpha \leqslant 1$.

A nontrivial solution of equation (1) is called oscillatory if it has arbitrarily large zero; and otherwise it is said to be non-oscillatory. Equation (1) is said to be oscillatory if its all solutions are oscillatory.

When $\alpha = 1$, we have the following second-order differential equation

$$(p(t)y'(t))' + q(t)y(t) = e(t), \tag{2}$$

Many papers involve the oscillation for equation (2) and other linear, nonlinear, damped, forced differential equations or Hamiltonian systems (see [3,8,9,10]) since the foundation work of Winter [10].

In this paper, we give a new type of oscillation criterion for equation (1), this oscillation criterion relies information of the coefficients on a sequence of interval on $[t_0,\infty)$, rather than on the whole interval.

1 Main results

For the sake of convenience, we give some definitions and properties of conformable fractional deriva

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tives and integral, which are important in the proofs of the main result. We begin with the following definitions and lemmas.

Definition 1 The left conformable fractional derivative starting from t_0 of a function $f:[t_0,\infty)\to R$ of order α with $0<\alpha\leqslant 1$ is defined by

$$(T_{t_0}^{\alpha}f)(t)=f^{(\alpha)}(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon(t-t_0)^{1-\alpha})-f(t)}{\varepsilon},$$

when $\alpha=1$, this derivative of $f(t)$ coincides with $f'(t)$. If $T_{t_0}^{\alpha}$ exists on (t_0,t_1) then

$$(T_{t_0}^{\alpha}f)(t)=\lim_{t\to t_0^+}f^{(\alpha)}(t).$$

Definition 2 Let $\alpha\in(0,1]$. The left conformable fractional integral of α starting at t_0 is defined by

$$(I_{t_0}^{\alpha}f)(t)=\int_{t_0}^{t}(s-t_0)^{\alpha-1}f(s)\mathrm{d}s=\int_{t_0}^{t}f(s)\mathrm{d}_{t_0}^{\alpha}.$$

If the conformable fractional integral of a given function exists, we call it is α-integrable.

Lemma 1 (see [1]) If $\alpha\in(0,1)$ and $f\in([t_0,\infty),R)$, then for all $t\geqslant t_0>0$, we have

$$I_{t_0}^{\alpha}(T_{t_0}^{\alpha}f)(t)=f(t)-f(t_0),\text{ and }T_{t_0}^{\alpha}(I_{t_0}^{\alpha}f)(t)=f(t).$$

Lemma 2 (see [4])

(1) $T_{t_0}^{\alpha}(af+bg)=aT_{t_0}^{\alpha}(f)+bT_{t_0}^{\alpha}(g)$ for all real constant a,b;

(2) $T_{t_0}^{\alpha}(fg)=fT_{t_0}^{\alpha}(g)+gT_{t_0}^{\alpha}(f)$;

(3) $T_{t_0}^{\alpha}(t^p)=pt^{p-\alpha}$, for all p;

(4) $T_{t_0}^{\alpha}(\frac{f}{g})=\frac{gT_{t_0}^{\alpha}(f)-fT_{t_0}^{\alpha}(g)}{g^2}$;

(5) $T_{t_0}^{\alpha}(c)=0$, where c is a constant.

Lemma 3 (chain rule, see [1,4,5]) Assume $f,g:(a,\infty)\to R$ be (left) α- differentiable functions, where $0<\alpha\leqslant 1$. Let $h(t)=f(g(t))$. Then $h(t)$ is (left) α- differentiable and for all t with $t\neq a$ and $g(t)\neq 0$ we have $(T_a^{\alpha}h)(t)=f'(g(t))(T_a^{\alpha}g)(t)$. If $t=a$ we have $(T_a^{\alpha}h)(t)=\lim_{t\to a^+}f'(g(t))(T_a^{\alpha}g)(t)$.

Lemma 4 (see [5]) Let $f,g:[t_0,t_1]\to R$ be two functions such f,g is α- differentiable. Then

$$\int_{t_0}^{t_1}f(s)g^{(\alpha)}(s)\mathrm{d}_{t_0}^{\alpha}s=f(s)g(s)\big|_{t_0}^{t_1}-\int_{t_0}^{t_1}g(s)f^{(\alpha)}(s)\mathrm{d}_{t_0}^{\alpha}s.$$

Lemma 5 (see [6]) Suppose X and Y are nonnegative. Then

$$\lambda XY^{\lambda-1}-X^{\lambda}\leqslant(\lambda-1)Y^{\lambda},\lambda>1,$$

where equality holds if and only if $X=Y$.

For $a,b\in R$ such that $a<b$, let

$$D(a,b)=\{u\in C^1[a,b]:u^2(t)>0,t\in(a,b),u(a)=u(b)=0\}.$$

Let $\rho\in C^1([t_0,\infty),R)$ be a positive function. For given $f\in C([t_0,\infty),R)$ as in [11], we define an integral operator A_a^b in terms of $H\in D(a,b)$ and ρ as

$$A_a^b(f;t)=\int_a^b H^2(t)\rho(t)f(t)\mathrm{d}_at,a\leqslant t\leqslant b.$$

By a direct calculation, we see that A_a^b has the following properties

$$A_a^b(\alpha_1f_1+\alpha_2f_2;t)=\alpha_1A_a^b(f_1;t)+\alpha_2A_a^b(f_2;t);\tag{3}$$

$$A_a^b(f;t)\geqslant 0\text{ whenever }f\geqslant 0;\tag{4}$$

$$A_a^b(f^{(\alpha)};t)=-A_a^b(\Big[2\frac{H^{(\alpha)}}{H}+\frac{\rho^{(\alpha)}}{\rho}\Big]f;t)\geqslant -A_a^b(\Big|2\frac{H^{(\alpha)}}{H}+\frac{\rho^{(\alpha)}}{\rho}\Big||f|;t),\tag{5}$$

where $f_1,f_2,f\in C([t_0,\infty),R),g\in C^1([t_0,\infty),R)$ and α_1,α_2 are real numbers.

Theorem 1 Assume $\frac{f(u)}{|u|^{\beta}\mathrm{sgn}u} \geqslant \delta > 0$ for some constant β with $\beta > 1$. Suppose further that for any $T \geqslant T_0$, there exist $T \leqslant s_1 < t_1 \leqslant s_2 < t_2$ such that $e(t) \leqslant 0$ for $t \in [s_1, t_1]$ and $e(t) \geqslant 0$ for $t \in [s_2, t_2]$. If there exist $H \in D(s_i, t_i)$ and a positive function $\rho \in C^1([t_0, \infty), R)$ such that

$$A_{s_i}^{t_i}(Q_e;t) > A_{s_i}^{t_i}\left(p\left|\frac{H^{(\alpha)}}{H} + \frac{\rho^{(\alpha)}}{\rho}\right|^2;t\right). \tag{6}$$

for $i = 1, 2$, where

$$Q_e(t) = \beta(\beta-1)^{\frac{1-\alpha}{\beta}}[\delta q(t)]^{\frac{1}{\beta}}|e(t)|^{\frac{\beta-1}{\beta}},$$

then equation (1) is oscillatory.

Proof Suppose to the contrary that there exists a nontrivial solution $y(t)$ of equation (1) which is not oscillatory. Without loss of generality, we may assume that $y(t) > 0$ on $[T_0, \infty)$ for some $T_0 \geqslant t_0$. Set

$$V(t) = \frac{p(t)y^{(\alpha)}(t)}{y(t)}. \tag{7}$$

Then differentiating (7) and making use of equation (1), it follows that for all $t \geqslant T_0$, we have

$$\begin{aligned} V^{(\alpha)}(t) &= \frac{y(t)[e(t) - q(t)f(y(t))]}{y^2(t)} - \frac{p(t)[y^{(\alpha)}(t)]^2}{y^2(t)} \\ &= \frac{-q(t)f(y(t))}{|y(t)|^{\beta-1}}|y(t)|^{\beta-1} + \frac{e(t)}{y(t)} - \frac{V^2(t)}{p(t)} \leqslant -\delta q(t)|y(t)|^{\beta-1} + \frac{e(t)}{y(t)} - \frac{V^2(t)}{p(t)}. \end{aligned} \tag{8}$$

By our assumption, we can choose $t_1 > s_1 \geqslant T_0$ so that $e(t) \leqslant 0$ on the interval $I_1 = [s_1, t_1]$. For given $t \in I_1$, set $F(x) = \delta q(t)x^{\beta-1} - \frac{e(t)}{x}$, we have $F'(x^*) = 0, F''(x^*) > 0$, where $x^* = \left[\frac{-e(t)}{(\beta-1)\delta q(t)}\right]^{\frac{1}{\beta}}$. So $F(x)$ attains its minimum at x^* and

$$F(x) \geqslant F(x^*) = Q_e(t), \tag{9}$$

So (8) and (9) imply that $V(t)$ satisfies

$$Q_e(t) \leqslant -V^{(\alpha)}(t) - \frac{V^2(t)}{p(t)},$$

Applying the operator $A_{s_1}^{t_1}$ to (8), using the fact that $H(s_1) = H(t_1) = 0$ and (5), we obtain

$$A_{s_1}^{t_1}(Q_e(t);t) \leqslant A_{s_1}^{t_1}\left(-V^{(\alpha)}(t) - \frac{V^2(t)}{p(t)};t\right) \leqslant A_{s_1}^{t_1}\left(2\left|\frac{H^{(\alpha)}}{H} + \frac{\rho^{(\alpha)}}{2\rho}\right||V(t)| - \frac{V^2(t)}{p(t)};t\right). \tag{10}$$

Let $X = \frac{|V|}{p^{\frac{1}{2}}}, \lambda = 2, Y = p^{\frac{1}{2}}\left|\frac{H^{(\alpha)}}{H} + \frac{\rho^{(\alpha)}}{2\rho}\right|$. By Lemma 5, we obtain

$$2|V|\left|\frac{H^{(\alpha)}}{H} + \frac{\rho^{(\alpha)}}{2\rho}\right| - \frac{V^2(t)}{p(t)} \leqslant p\left|\frac{H^{(\alpha)}}{H} + \frac{\rho^{(\alpha)}}{2\rho}\right|^2, \tag{11}$$

Then in view of (10), (11) and the properties (3) as well as (4), we see that

$$A_{s_1}^{t_1}(Q_e;t) > A_{s_1}^{t_1}\left(p\left|\frac{H^{(\alpha)}}{H} + \frac{\rho^{(\alpha)}}{2\rho}\right|^2;t\right),$$

which is contrary to (6). On the other hand, if $y(t)$ is a negative solution for $t \geqslant T_0 > t_0$, we define the Riccati transformation (7) to yield (8). In this case, we choose $t_2 > s_2 \geqslant T_0$ so that $e(t) \geqslant 0$ on the interval $I_2 = [s_2, t_2]$. For given $t \in I_2$, set $F(x) = \delta q(t)x^{\beta-1} - \frac{e(t)}{x}$, we have $F(x) \geqslant F(x^*) = Q_e(t)$. The remaining proof is similar hence we omit the details. This completes our proof.

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非线性整合分数阶微分方程的振动性判据

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摘 要 本文给出了一类二阶非线性整合分数阶微分方程的振动性判据,这一判据只依赖于系数函数在正半轴的一列子区间上的性质.

关键词 整合分数阶微分方程;振动性;Riccati 变换