**文章编号** 1672-6634(2019)01-0030-08 **DOI** 10.19728/j.issn1672-6634.2019.01.005

# Hamiltonian Structure and Algebro-geometric Solutions of the Nonlinear Schrödinger-MKdV Equations

YUE Chao

(School of Medical Information Engineering, Taishan Medical University, Taian 271016, China)

**Abstract** A $3\times 3$ isospectral Lax matrix is introduced from which a nonlinear Schrödinger-MKdV(NLS-MKdV)hierarchy is obtained, whose Hamiltonian structure is presented by use of the trace identity. For the convenience of obtaining algebro-geometric solutions, we transform the $3\times 3$ matrix isospectral problem into an equivalent $2\times 2$ one, in terms of Riemann theta functions, the algebro-geometric solutions of the coupled NLS equation and coupled MKdV equation reduced from the NLS-MKdV hierarchy are constructed.

**Key words** the trace identity; Hamiltonian structure; algebro-geometric solution; Riemann theta function

**CLC** O175.2 **Document code** A

## 0 Introduction

Searching for the exact solutions of nonlinear equations has been important and interesting in the areas of the mathematics and physics, and several systematic methods have been developed to obtain explicit solutions of soliton equations, for instance, the inverse scattering method[1], Darboux and Bäcklund transformations[2], Hirota's bilinear method[2-4], Lie symmetry analysis etc[5-12]. The algebraic-geometric method was first developed by Matveev et al. as an analog of the inverse scattering theory. As a degenerated case of the algebro-geometric solutions, the multi-soliton solution and periodic solution in elliptic function type may be worked out. A systematic approach, proposed by Gesztesy and Holden to construct algebro-geometric solutions for integrable equations, has been extended to the whole $(1+1)$ dimensional integrable hierarchy, such as the AKNS hierarchy, the Camassa-Holm hierarchy etc[13]. Recently, Fan etc. investigated algebro-geometric solutions for the Gerdjikov-Ivanov hierarchy, the Hunter-Saxton hierarchy and so on[14-17].

In this paper, we first use a $3\times 3$ isospectral Lax matrix to obtain a NLS-MKdV hierarchy by use of the Tu scheme[18-23], which can reduce to the coupled NLS equation and coupled MKdV equation and whose Hamiltonian structure can be generated by applying the trace identity. As we know, constructing algebro-geometric solutions associated with the $3\times 3$ matrix isospectral problem is more complicated than that related to the $2\times 2$ case, hence we transform the above $3\times 3$ matrix isospectral problem into an equivalent $2\times 2$ one, by using Riemann theta functions, the algebro-geometric solutions of the coupled NLS equation and coupled MKdV equation are obtained easily.

## 1 The NLS-MKdV hierarchy and its Hamiltonian structure

Consider the $3\times 3$ isospectral Lax matrix

**收稿日期**:2018-07-12

**基金项目**:国家自然科学基金项目(11805114);山东省自然科学基金资助课题(ZR2016AL04,ZR2017MF039,ZR2016FL05,ZR2012AM021);泰山医学院高层次课题培育计划项目(2015GCC07)资助

**通讯作者**:岳超,男,汉族,博士,副教授,研究方向:孤立子理论与可积系统,E-mail:yuechao_71@163.com.

$$\boldsymbol{U}=\frac{\sqrt{2}}{2}\begin{pmatrix}\sqrt{2}\lambda & q+r & 0\\ q-r & 0 & q+r\\ 0 & q-r & -\sqrt{2}\lambda\end{pmatrix},\boldsymbol{V}=\frac{\sqrt{2}}{2}\sum_{m\geqslant 0}\begin{pmatrix}\sqrt{2}a_m & 2b_m & 0\\ 2c_m & 0 & 2b_m\\ 0 & 2c_m & -\sqrt{2}a_m\end{pmatrix}\lambda^{-m}, \tag{1}$$

solving the equation $\boldsymbol{V}_x=[\boldsymbol{U},\boldsymbol{V}]$, leads to (2)

$$\begin{cases} a_{mx}=(q+r)c_m+(r-q)b_m,\\ b_{m+1}=b_{mx}+\dfrac{1}{2}(q+r)a_m,\\ c_{m+1}=-c_{mx}+\dfrac{1}{2}(q-r)a_m,\\ a_0=2\beta,b_0=c_0=a_1=0,b_1=\beta(q+r),c_1=\beta(q-r),b_2=\beta(q_x-r_x),\\ c_2=\beta(r_x-q_x),a_2=\beta(r^2-q^2),b_3=\beta(q_{xx}+r_{xx})+\dfrac{1}{2}\beta(q+r)(r^2+q^2),\\ c_3=\beta(q_{xx}-r_{xx})+\dfrac{1}{2}\beta(q-r)(r^2-q^2),a_3=2\beta(rq_x-qr_x),\\ b_4=\beta(q_{xxx}+r_{xxx})+\dfrac{1}{2}\beta[(q+r)(r^2-q^2)]_x+\beta(q+r)(rq_x-qr_x),\\ c_4=\beta(r_{xxx}-q_{xxx})-\dfrac{1}{2}\beta[(q-r)(r^2-q^2)]_x+\beta(q-r)(rq_x-qr_x),\cdots. \end{cases} \tag{3}$$

Note

$$\boldsymbol{V}_{+}^{(n)}=\frac{\sqrt{2}}{2}\sum_{m=0}^{n}\begin{pmatrix}\sqrt{2}a_m & 2b_m & 0\\ 2c_m & 0 & 2b_m\\ 0 & 2c_m & -\sqrt{2}a_m\end{pmatrix}\lambda^{n-m},$$

a direct calculation may show that the compatibility conditions of the Lax pairs $\boldsymbol{\varphi}_x=U\boldsymbol{\varphi},\boldsymbol{\varphi}_{t_n}=\boldsymbol{V}^{(n)},\boldsymbol{V}^{(n)}=\boldsymbol{V}_{+}^{(n)}$, engenders the integrable hierarchy,

$$\boldsymbol{u}_{t_n}=\begin{pmatrix}q\\ r\end{pmatrix}_{t_n}=\begin{pmatrix}b_{n+1}-c_{n+1}\\ b_{n+1}+c_{n+1}\end{pmatrix}=\begin{pmatrix}0 & -\dfrac{1}{2}\\ \dfrac{1}{2} & 0\end{pmatrix}\begin{pmatrix}2(b_{n+1}+c_{n+1})\\ 2(c_{n+1}-b_{n+1})\end{pmatrix}=\boldsymbol{J}\boldsymbol{P}_{n+1}, \tag{4}$$

where $\boldsymbol{J}$ is a Hamiltonian operator, from (3), we obtain a recurrence operator

$$\boldsymbol{L}=\begin{pmatrix}q\partial^{-1}r & q\partial^{-1}q-\partial\\ \partial+r\partial^{-1}r & r\partial^{-1}q\end{pmatrix}.$$

Therefore, expression (4) can be written as

$$\boldsymbol{u}_{t_n}=\begin{pmatrix}q\\ r\end{pmatrix}_{t_n}=\boldsymbol{J}\boldsymbol{L}^n\begin{pmatrix}4\beta q\\ -4\beta r\end{pmatrix}. \tag{5}$$

**Reduction case** 1 When $n=2$, the system (5) reduces to the following coupled NLS equation

$$\begin{cases} q_{t_2}=2\beta r_{xx}+\beta r(r^2-q^2),\\ r_{t_2}=2\beta q_{xx}+\beta q(r^2-q^2). \end{cases} \tag{6}$$

Taking $\beta=i,R=q+r=i\bar{q}+r$, we obtain the NLS equation

$$iR_{t_2}+2R_{xx}+R|R|^2=0.$$

**Reduction case** 2 When taking $n=3$ in (5), we have the coupled MKdV equation

$$\begin{cases} q_{t_3}=2\beta q_{xxx}+3\beta q_x(r^2-q^2),\\ r_{t_3}=2\beta q_{xx}+3\beta r_x(r^2-q^2). \end{cases} \tag{7}$$

Taking $\beta=1,q=0$, we get the MKdV equation $r_{t_3}-3r^2r_x-2r_{xxx}=0$. Hence we call the system (5) NLS-MKdV hierarchy.

A direct computation yields

$$\left\langle V,\frac{\partial\boldsymbol{U}}{\partial\boldsymbol{q}}\right\rangle=2b+2c,\left\langle\boldsymbol{V},\frac{\partial\boldsymbol{U}}{\partial\boldsymbol{r}}\right\rangle=2c-2b,\frac{\partial\boldsymbol{U}}{\partial\lambda}=2a,$$

substituting the above equations into the trace identity [18], we get

$$\frac{\delta}{\delta u}(2a)=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}2b+2c\\2c-2b\end{pmatrix}. \tag{8}$$

Comparing the coefficients of $\lambda^{-n-1}$ on both sides in (8) leads to

$$\frac{\delta}{\delta u}(2a_{n+1})=(\gamma-n)P_n,$$

it is easy to find that $\gamma=0$, then we have

$$P_n=\frac{\delta}{\delta u}\left(\frac{-2a_{n+1}}{n}\right)=\frac{\delta H_n}{\delta u}.$$

Hence, we obtain the Hamiltonian structure of (5)

$$\boldsymbol{u}_{t_n}=\begin{pmatrix}q\\r\end{pmatrix}_{t_n}=\boldsymbol{J}\frac{\delta H_{n+1}}{\delta u}. \tag{9}$$

It is easy to verify that $\boldsymbol{JL}=\boldsymbol{L}^*\boldsymbol{J}$, so the NLS-MKdV hierarchy is integrable in Liouville sense. In the following section, we are interested in constructing algebro-geometric solutions of the coupled NLS equation (6) and the coupled MKdV equation (7).

## 2 Algebro-geometric solutions of the coupled NLS equation (6) and the coupled MKdV equation (7)

For calculation convenience, we transform $3\times3$ matrix isospectral problem (1) into an equivalent $2\times2$ one,

$$\boldsymbol{\psi}_x=\boldsymbol{U}'\boldsymbol{\psi},\ \boldsymbol{\psi}_t=\boldsymbol{V}'\boldsymbol{\psi},\ \boldsymbol{U}'=\frac{1}{2}\begin{pmatrix}\lambda & q+r\\ q-r & -\lambda\end{pmatrix},$$

$$\boldsymbol{V}'^{(m)}=\begin{pmatrix}A^{(m)} & B^{(m)}\\ C^{(m)} & -A^{(m)}\end{pmatrix}=\frac{1}{2}\sum_{j=0}^{m}\begin{pmatrix}a_j & 2b_j\\ 2c_j & -a_j\end{pmatrix}\lambda^{m-j}, \tag{10}$$

which can also generate equations (6) and (7). We consider the Lenard gradient sequence $\{\boldsymbol{S}_j\}_{j=0}^{\infty}$ by the recursion relation

$$\boldsymbol{KS}_{j-1}=\boldsymbol{JS}_j,\boldsymbol{S}_j|_{(q,r)}=0,\boldsymbol{S}_0=(2\beta,0,0)^{\mathrm{T}}, \tag{11}$$

where

$$\boldsymbol{K}=\begin{pmatrix}\partial & q+r & -q-r\\ \frac{1}{2}(q+r) & \partial & 0\\ \frac{1}{2}(q-r) & 0 & -\partial\end{pmatrix},\boldsymbol{J}=\begin{pmatrix}\partial & q+r & -q-r\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

It is easy to find that $\boldsymbol{S}_j$ is uniquely determined by the recursion relation (10). Here the condition $S_j|_{(q,r)}=0$ is used to select the integration constant to be zero. A direct computation shows from (11) that

$$\boldsymbol{S}_1=\begin{pmatrix}0\\ \beta(q+r)\\ \beta(q-r)\end{pmatrix},\boldsymbol{S}_2=\begin{pmatrix}\beta(r^2-q^2)\\ \beta(q_x+r_x)\\ \beta(r_x-q_x)\end{pmatrix},\cdots.$$

We suppose (10) has two basic solutions $\boldsymbol{X}=(X_1,X_2)^{\mathrm{T}}$ and $\boldsymbol{Y}=(Y_1,Y_2)^{\mathrm{T}}$, then

$$\boldsymbol{W}=\frac{1}{2}(XY^{\mathrm{T}}+YX^{\mathrm{T}})\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}g & f\\ h & -g\end{pmatrix},$$

satisfies the Lax equation

$$\boldsymbol{W}_x=[\boldsymbol{U},\boldsymbol{W}],\ \boldsymbol{W}_{t_m}=[\boldsymbol{V}^{(m)},\boldsymbol{W}], \tag{12}$$

which implies that the function $\det(\boldsymbol{W})$ is a constant independent of $x$ and $t_m$. From (12), we get

$$2g_x=(q+r)h+(r-q)f,f_x=\lambda f-(q+r)g,h_x=-\lambda h+(q-r)g, \tag{13}$$

and

$$g_{t_m}=B^{(m)}h-C^{(m)}f,\ f_{t_m}=2A^{(m)}f-2B^{(m)}g,\ h_{t_m}=2C^{(m)}g-2A^{(m)}h, \tag{14}$$

where

$$g=\frac{1}{2}\sum_{j=0}^{N+1}a_j\lambda^{N+1-j},\ f=\sum_{j=0}^{N+1}b_j\lambda^{N+1-j},\ h=\sum_{j=0}^{N+1}c_j\lambda^{N+1-j},\tag{15}$$

and $N$ is an arbitrary positive integer value. Substituting (15) into (13) gives

$$\boldsymbol{KQ}_{j-1}=\boldsymbol{JQ}_j,\ \boldsymbol{JQ}_0=0\ ,\ \boldsymbol{KQ}_N=0,\ \boldsymbol{Q}_j=(a_j,b_j,c_j)^{\mathrm{T}}.\tag{16}$$

It is clear to find $JQ_0=0$ that has the general solution

$$\boldsymbol{Q}_0=\alpha_0\boldsymbol{S}_0=\alpha_0(2\beta,0,0)^{\mathrm{T}},\tag{17}$$

from (11) and (16), we have

$$\boldsymbol{Q}_K=\sum_{j=0}^{k}\alpha_j\boldsymbol{S}_{k-j},0\leqslant k\leqslant N+1,\tag{18}$$

where $a_0,\cdots,a_{k+1}$ are integral constants. Substituting Eq. (18) into Eq. (16) yields the following certain stationary evolution equation

$$\alpha_0\ \tilde{X}_N+\cdots+\alpha_N\ \tilde{X}_0=0,\tag{19}$$

without loss of generality we set $a_0=1$, from Eqs. (15), (16)and (18), we then have

$$\boldsymbol{Q}_1=\begin{pmatrix}2\beta\alpha_1\\ \beta(q+r)\\ \beta(q-r)\end{pmatrix},\boldsymbol{Q}_2=\begin{pmatrix}\beta(r^2-q^2)+2\beta\alpha_2\\ \beta(q_x+r_x)+\beta\alpha_1(q+r)\\ \beta(r_x-q_x)+\beta\alpha_1(q-r)\end{pmatrix},$$

$$\boldsymbol{Q}_3=\begin{pmatrix}2\beta(rq_x-qr_x)+\beta\alpha_1(r^2-q^2)+2\beta\alpha_3\\ \beta(q_{xx}+r_{xx})+\frac{1}{2}\beta(q+r)(r^2-q^2)+\beta\alpha_1(q_x+r_x)+\beta\alpha_2(q+r)\\ \beta(q_{xx}-r_{xx})+\frac{1}{2}\beta(q-r)(r^2-q^2)+\beta\alpha_1(r_x-q_x)+\beta\alpha_2(q-r)\end{pmatrix}.\tag{20}$$

By applying (15) we can rewrite $f$ and $h$ as the following finite products

$$f=\beta(q+r)\prod_{j=1}^{N}(\lambda-\mu_j),\ h=\beta(q-r)\prod_{j=1}^{N}(\lambda-\nu_j).\tag{21}$$

By comparing the coefficients of $\lambda^{N-1},\lambda^{N-2}$ and combining Eqs. (15) and (21), we get

$$\frac{q_x+r_x}{q+r}+\alpha_1=-\sum_{j=1}^{N}\mu_j,\frac{r_x-q_x}{q-r}+\alpha_1=-\sum_{j=1}^{N}\nu_j,\tag{22}$$

$$\frac{q_{xx}+r_{xx}}{q+r}+\frac{1}{2}(r^2-q^2)+\alpha_1\ \frac{q_x+r_x}{q+r}+\alpha_2=\sum_{j<k}\mu_j\mu_k,\frac{q_{xx}-r_{xx}}{q-r}+\frac{1}{2}(r^2-q^2)+\alpha_1\ \frac{r_x-q_x}{q-r}+\alpha_2=\sum_{j<k}\nu_j\nu_k,\tag{23}$$

since det $(\boldsymbol{W})$ is a $(2N+2)$ th-order polynomial in $\lambda$ with constants, we have

$$-\det(\boldsymbol{W})=g^2+fh=4\beta^2\prod_{j=1}^{2N+2}(\lambda-\lambda_j)=4\beta^2R(\lambda).\tag{24}$$

Substituting Eq. (15) into Eq. (24) and comparing the coefficient of $\lambda^{2N+2},\lambda^{2N}$, gives

$$2g_0g_1+f_0h_1+f_1h_0=-\sum_{j=1}^{2N+2}\lambda_j,g_1^2+2g_0g_2+f_1h_1=\sum_{j<k}\lambda_j\lambda_k,\tag{25}$$

hence we obtain

$$\alpha_1=\frac{-1}{2\beta^2}\sum_{j=1}^{2N+2}\lambda_j,\beta_2=\frac{1}{2\beta^2}\Big[\sum_{j<k}\lambda_j\lambda_k-\frac{1}{4\beta^2}\Big(\sum_{j=1}^{2N+2}\lambda_j\Big)^2\Big].\tag{26}$$

From (24)we have

$$g|_{\lambda=\mu_k}=2|\beta|\sqrt{R(\mu_k)},g|_{\lambda=v_k}=2|\beta|\sqrt{R(v_k)}.\tag{27}$$

Again utilizing (13) and (21), we find

$$f_x|_{\lambda=\mu_k}=-\beta(q+r)\mu_{kx}\prod_{j=1,j\neq k}^{N}(\mu_k-\mu_j)=-(q+r)g|_{\lambda=\mu_k},$$

$$h_x|_{\lambda=v_k}=-\beta(r-q)v_{kx}\prod_{j=1,j\neq k}^{N}(v_k-v_j)=-(q-r)g|_{\lambda=\nu_k},\tag{28}$$

which together with (27) leads to

$$\mu_{kx}=\frac{2|\beta|\sqrt{R(\mu_k)}}{\beta\prod_{j=1,j\neq k}^{N}(\mu_k-\mu_j)},v_{kx}=-\frac{2|\beta|\sqrt{R(v_k)}}{\beta\prod_{j=1,j\neq k}^{N}(v_k-v_j)}.\tag{29}$$

Similarly, by use of (14), (21) and (27), we obtain

$$\mu_{kt_m}=\frac{4|\beta|\sqrt{R(\mu_k)}B^{(m)}}{\beta(q+r)\prod_{j=1,j\neq k}^{N}(\mu_k-\mu_j)},v_{kt_m}=-\frac{4|\beta|\sqrt{R(v_k)}C^{(m)}}{\beta(r-q)\prod_{j=1,j\neq k}^{N}(v_k-v_j)},\tag{30}$$

thus

$$\mu_{kt_2}=\frac{4|\beta|(\mu_k-\sum_{j=1}^{N}\mu_j-\alpha_1)\sqrt{R(\mu_k)}}{\prod_{j=1,j\neq k}^{N}(\mu_k-\mu_j)},v_{kt_2}=\frac{4|\beta|(v_k-\sum_{j=1}^{N}v_j-\alpha_1)\sqrt{R(v_k)}}{\prod_{j=1,j\neq k}^{N}(v_k-v_j)},\tag{31}$$

$$\mu_{kt_3}=\frac{4|\beta|[(\mu_k^2-\sum_{j=1}^{N}\mu_j+\alpha_1)\mu_k+\sum_{j<k}\mu_j\mu_k+(\sum_{j=1}^{N}\mu_j+\alpha_1)\alpha_1-\alpha_2]+\sqrt{R(\mu_k)}}{\prod_{j=1,j\neq k}^{N}(\mu_k-\mu_j)},$$

$$v_{kt_3}=\frac{-4|\beta|[(v_k^2-\sum_{j=1}^{N}v_j+\alpha_1)v_k+\sum_{j<k}v_jv_k+(\sum_{j=1}^{N}v_j+\alpha_1)\alpha_1-\alpha_2]+\sqrt{R(v_k)}}{\prod_{j=1,j\neq k}^{N}(v_k-v_j)},\tag{32}$$

hence let $\mu_k(x,t_m),v_k(x,t_m)$ be distinct solutions of the ordinary differential Eqs. (29) and(30), then $(q,r)$ determined by (22) is a solution of Eq. (6) with $n=m=2$ or a solution of Eq. (7) with $n=m=3$.

Based on the form of the function det $(\boldsymbol{W})$ in Eq. (24), we introduce the hyperelliptic Riemann surface

$$\Gamma:\xi^2=R(\lambda),R(\lambda)=\prod_{j=1}^{2N+2}(\lambda-\lambda_1),\lambda_{2N+2}=0,$$

with genus $g=N$. For the same , there are two points $(\lambda,\sqrt{R(\lambda)})$ and $(\lambda,-\sqrt{R(\lambda)})$ on different sheets of $\Gamma$. Since $R(\lambda)$ is a polynomial of order $2N+2$ in terms of $\lambda$, there are two infinite points $\infty_1$ and $\infty_2$ which are not branch points of $\Gamma$. On $\Gamma$ we fix a set of regular cycle paths: $a_1,a_2,...,a_N;b_1,b_2,...,b_N$, which are independent and have the intersection numbers as follows

$$a_k\circ a_j=b_k\circ b_j=0,a_k\circ b_j=\delta_{kj},k,j=1,\cdots,N.$$

The holomorphic differentials on $\Gamma$ are chosen to be

$$\tilde{\omega}_l=\frac{\lambda^{l-1}\mathrm{d}\lambda}{\sqrt{R(\lambda)}},l=1,2,\cdots,N.$$

Let $A_{kj}=\int_{a_j}\tilde{\omega}_k,B_{kj}=\int_{b_j}\tilde{\omega}_k$. The $N\times N$ matrices $\boldsymbol{A}=(A_{kj})$ and $\boldsymbol{B}=(B_{kj})$ are invertible. Define matrices $\boldsymbol{C}$ and $\boldsymbol{\tau}$ by $\boldsymbol{C}=\boldsymbol{A}^{-1},\boldsymbol{\tau}=\boldsymbol{A}^{-1}\boldsymbol{B}$. The matrix $\tau$ can be shown to be symmetric and has a positive definite imaginary part. We normalize $\widehat{\omega}_j$ into the following new basis

$$\omega_j=\sum_{l=1}^{N}c_{jl}\tilde{\omega}_l,l=1,2,\cdots,N.$$

Then we find

$$\int_{a_k}\omega_j=\sum_{l=1}^{N}C_{jl}\int_{a_k}\tilde{\omega}_l=\sum_{l=1}^{N}C_{jl}A_{lk}=\delta_{jk},\int_{b_k}\omega_j=\tau_{jk}.$$

For a fixed point $p_0$ , the Abel-Jacobi coordinate are given as follows

$$\rho_1{}^{(j)}(x,t_m)=\sum_{k=1}^{N}\int_{\rho_0}^{uk(x,t_m)}\omega_j=\sum_{k=1}^{N}\sum_{l=1}^{N}\int_{\rho_0}^{\mu_k}C_{jl}\frac{\lambda^{l-1}\mathrm{d}\lambda}{\sqrt{R(\lambda)}},\tag{33}$$

$$\rho_2^{(j)}(x,t_m)=\sum_{k=1}^{N}\int_{p_0}^{\nu_k(x,t_m)}\omega_j=\sum_{k=1}^{N}\sum_{l=1}^{N}\int_{p_0}^{\nu_k}C_{jl}\frac{\lambda^{l-1}\mathrm{d}\lambda}{R(\lambda)}.\tag{34}$$

By using (33) and the first expression of (29), we have

$$\partial_x p_1^{(j)}=\sum_{k=1}^{N}\sum_{l=1}^{N}C_{jl}\frac{\mu_k^{l-1}\mu_{kx}}{\sqrt{R(\lambda)}}=\sum_{k=1}^{N}\sum_{l=1}^{N}\frac{2\alpha\mu_k^{l-1}C_{jl}}{\Pi_{j=1,j\neq k}^{N}(\mu_k-\mu_j)},$$

which gives

$$\partial_x p_1^{(j)}=\frac{2|\beta|}{\beta}C_{jN}=\Omega_1^{(j)}=,j=1,2,\cdots,N,$$

by use of the following equality

$$\sum_{k=1}^{N}\frac{\mu_k^{l-1}}{\Pi_{j=1,j\neq k}(\mu_k-\mu_j)}=\delta_{jN},l=1,2,\cdots,N.$$

In a similar way, we get from (29)-(34)

$$\partial_x \rho_2^{(j)} = -\Omega_1^{(j)}, \partial_{t_2 \rho_1}{}^{(j)} = \partial_{t_2 \rho_2}{}^{(j)} = \Omega_2^{(j)} = 4|\beta|(C_{jN-1} - \alpha_1 C_{jN}),$$

$$\partial_{t_3 \rho_1}{}^{(j)} = \partial_{t_3 \rho_2}{}^{(j)} = \Omega_3^{(j)} = 4|\beta|(C_{jN-2} - \alpha_1 C_{jN-1} + \alpha_1^2 C_{jN} - \alpha_2 C_{jN}), j=1,2,\cdots,N.$$

Based on the above results we have the following

$$\rho_1 = \Omega_1 x + \Omega_m t_m + \gamma_1, \rho_2 = -\Omega_1 x - \Omega_m t_m + \gamma_2,$$

where

$$\Omega_m = (\Omega_m^{(1)}, \Omega_m^{(2)}, \cdots, \Omega_m^{(N)})^{\mathrm{T}}, m=2,3,$$

$$\gamma_1^{(j)} = \sum_{k=1}^N \int_{p_0}^{\mu_k(0,0)} \omega_j, \gamma_2^{(j)} = \sum_{k=1}^N \int_{p_0}^{v_k(0,0)} \omega_j.$$

We define an Abel map on $\Gamma$ as follows

$$A(p) = \int_{p_0}^p \omega, \omega = (\omega_1, \cdots, \omega_N)^{\mathrm{T}}, A(\sum n_k p_k) = \sum n_k A(p_k).$$

Consider two special divisors $m$ ($m=1,2$); then we get

$$A(\sum_{k=1}^N p_1^{(k)}) = \sum_{k=1}^N A(p_1^{(k)}) = \sum_{k=1}^N \int_{p_0}^{v_k} \omega = \rho_1,$$

$$A(\sum_{k=1}^N p_2^{(k)}) = \sum_{k=1}^N A(p_2^{(k)}) = \sum_{k=1}^N \int_{p_0}^{v_k} \omega = \rho_2,$$

$$p_1^{(k)} = (\mu_k, \xi(\mu_k)) \text{ and } p_2^{(k)} = (v_k, \xi(v_k)).$$

We define the Riemann theta function of $\Gamma$ as

$$\theta(\zeta) = \sum_{z \in \mathbf{Z}^N} \exp(\pi i \langle \tau z, z \rangle + 2\pi i \langle \zeta, z \rangle), \zeta \in \mathbf{C}^N,$$

in which $\boldsymbol{\zeta} = (\zeta, \cdots, \zeta_N)^{\mathrm{T}}, \langle \zeta, z \rangle = \sum_{k=1}^N \zeta_j z_j$. In terms of the Riemann theorem in algebraic geometry, there exist two constant vectors $M_1, M_2 \in \mathbf{C}^N$ such that

$F_1 = \theta(A(p) - \rho_1 - M_1)$ has exactly $N$ zeros at $\lambda = \mu_1, \ldots, \mu_N$; and

$F_2 = \theta(A(p) - \rho_2 - M_2)$ has exactly $N$ zeros at $\lambda = \nu_1, \ldots, \nu_N$.

In order to make these functions single valued, the surface $\Gamma$ is cut along all $a_k, b_k$ to form a simply connected region, whose boundary is denoted by $\gamma$. Notice the fact that the integrals

$$\frac{1}{2\pi j} \int_\gamma \lambda \, \mathrm{dln} F_m = I(\Gamma), \quad m=1,2,$$

are constants independent of $\rho_1$ and $\rho_2$ with

$$I = I(\Gamma) = \sum_{j=1}^N \int_{a_j} \lambda \omega_j.$$

Applying the residue theorem, we have

$$\sum_{j=1}^N \mu_j = I - \sum_{s=1}^2 Res_{\lambda=\infty_s}, \lambda \, \mathrm{dln} F_1(\lambda), \tag{35}$$

$$\sum_{j=1}^N v_j = I - \sum_{s=1}^2 Res_{\lambda=\infty_s}, \lambda \, \mathrm{dln} F_2(\lambda). \tag{36}$$

In order to compute the residues in (35) and (36), we first introduce local coordinates $z = \lambda^{-1}$ at infinity. Then the hyperelliptic curve $\xi^2 = R(\lambda)$ in the neighborhood of infinity can be expressed as $\tilde{\xi}^2 = \tilde{R}(z)$ with $\tilde{\xi} = z^{N+1}\xi, \tilde{R}(z) = \Pi_{j=1}^{2N+2}(1-\lambda_j z)$ and $\infty_s = (z, (-1)^{s-1}\sqrt{\tilde{R}(\lambda)})|_{z=0} = (0, (-1)^{s-1}), s=1,2$. It is easy to see that

$$A(p)_j = -\int_{\infty_s}^{p_0} + \int_{\infty_s}^p \omega_j = -\eta_s^{(j)} + \sum_{l=1}^N C_{jl} \int_\infty^p \frac{\lambda^{l-1} \mathrm{d}\lambda}{(-1)^{s-1}\sqrt{R(\lambda)}}$$

$$= -\eta_s^{(j)} - (-1)^{s-1} \sum_{l=1}^N C_{jl} \int_0^z \frac{z^{N-r} \mathrm{d}z}{\sqrt{\tilde{R}(z)}} = -\eta_s^{(j)} - (-1)^{s-1}[C_{jN} z + o(z^2)].$$

Since the Riemann theta function is an even function, $F_m(\lambda)$ can be written as

$$F_m(z^{-1}) = \theta(\cdots, p_m^{(j)} + M_m^{(j)} + \eta_s^{(j)} + (-1)^{s-1} C_{j\mathbf{N}^z} + o(z^2), \cdots)$$

$$= \theta_s^{(m)} + z(-1)^{s-1} \sum_{j=1}^N C_{jN} D_j \theta_s^{(m)} + o(z^2), \tag{37}$$

where $\theta_s^{(m)} = \theta(\rho_m + M_m + \eta_s) = \theta(\cdots, \rho_m^{(j)} + M_m^{(j)} + \eta_s^{(j)}, \cdots)$, $D_j$ signifies its derivative with respect to the $j$ th argument of $\theta_s^{(m)}$. It is easy to compute that

$$\partial_x\theta_s^{(m)}=\sum_{j=1}^{N}\frac{2|\beta|}{\beta}C_{jN}D_j\theta_s^{(m)}. \tag{38}$$

Substituting Eq. (38) into Eq. (37), we arrive at

$$F_m(z^{-1})=\theta_s^{(m)}+\frac{\beta z}{2|\beta|}(-1)^{s-1}\partial_x\theta_s^{(m)}+o(z^2),$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln F_m(z^{-1})=\frac{\beta}{2|\beta|}(-1)^{s-1}\partial_x\ln\theta_s^{(m)}+o(z).$$

Hence we have

$$\mathrm{Res}_{\lambda=\infty_s}\lambda\,\mathrm{d}\ln F_m(\lambda)=\mathrm{Res}_{z=0}z^{-1}\mathrm{d}\ln F_m(z^{-1})=\frac{\beta}{2|\beta|}(-1)^{s-1}\partial_x\ln\theta_s^{(m)},s=1,2,m=1,2, \tag{39}$$

where

$$\theta_s^{(1)}=\theta(\Omega_1x+\Omega_mt_m+\pi_s),\theta_s^{(2)}=\theta(-\Omega_1x-\Omega_mt_m+\eta_s),$$

and $\pi_s$ and $\eta_s$ are constants.

From Eqs. (35), (36) and (39), we get

$$\sum_{j=1}^{N}\mu_j=I+\frac{\beta}{2\beta}\partial_x\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}},\sum_{j=1}^{N}v_j=I+\frac{\beta}{2\beta}\partial_x\ln\frac{\theta_1^{(2)}}{\theta_2^{(2)}}. \tag{40}$$

Substituting Eq. (40) into Eqs. (22), we finally obtain the following algebraic-geometric solutions of Eq. (6) with $n=m=2$ or of Eq. (7) with $n=m=3$,

$$q=\frac{\beta}{4|\beta|}\left\{\frac{\theta_1^{(2)}}{\theta_2^{(2)}}\exp[(I+\alpha_1)x+r_0(t_m)]-\frac{\theta_2^{(1)}}{\theta_1^{(1)}}\exp[(-I-\alpha_1)x+q_0(t_m)]\right\}$$
$$=\frac{\beta}{4|\beta|}\left\{\frac{\theta(-\Omega_1x-\Omega_mt_m+\eta_1)}{\theta(-\Omega_1x-\Omega_mt_m+\eta_2)}\exp\left[\left(I-\frac{1}{2\beta^2}\sum_{j=1}^{2N+2}\lambda_j\right)x+r_0(t_m)\right]-\frac{\theta(\Omega_1x+\Omega_mt_m+\pi_2)}{\theta(\Omega_1x+\Omega_mt_m+\pi_1)}\exp\left[\left(\frac{1}{2\beta^2}\sum_{j=1}^{2N+2}\lambda_j-I\right)x+q_0(t_m)\right]\right\},$$
$$r=-\frac{\beta}{4|\beta|}\left\{\frac{\theta_1^{(2)}}{\theta_2^{(2)}}\exp[(I+\alpha_1)x+r_0(t_m)]+\frac{\theta_2^{(1)}}{\theta_1^{(1)}}\exp[(-I-\alpha_1)x+q_0(t_m)]\right\}$$
$$=\frac{\beta}{4|\beta|}\left\{\frac{\theta(-\Omega_1x-\Omega_mt_m+\eta_1)}{\theta(-\Omega_1x-\Omega_mt_m+\eta_2)}\exp\left[\left(I-\frac{1}{2\beta^2}\sum_{j=1}^{2N+2}\lambda_j\right)x+r_0(t_m)\right]+\frac{\theta(\Omega_1x+\Omega_mt_m+\pi_2)}{\theta(\Omega_1x+\Omega_mt_m+\pi_1)}\exp\left[\left(\frac{1}{2\beta^2}\sum_{j=1}^{2N+2}\lambda_j-I\right)x+q_0(t_m)\right]\right\},$$

where $q_0(t_m)$ and $r_0(t_m)$ are two arbitrary complex functions about variable $t_m$.

# 3 Conclusions

We obtained a nonlinear NLS-MKdV hierarchy and its Hamiltonian structure by use of the Tu scheme, furthermore, for the convenience of obtaining algebro-geometric solutions, we transform the $3\times3$ matrix isospectral problem into an equivalent $2\times2$ one, then the algebro-geometric solutions of the coupled NLS equation and coupled MKdV equation are constructed in terms of Riemann theta functions easily.

## References


[1] Gardner C S, Greene J M, Kruskal M D, et al. Method for solving the Korteweg-de Vries equation[J]. Phys Rev Lett, 1967,19: 1095-1097.

[2] Li Y S. Soliton and Integrable Systems, Advanced Series in Nonlinear Science[M]. Shanghai: Shanghai Scientific and Technological Education Publishing House, 1999.

[3] Hirota R, Satsuma J. A variety of nonlinear network equations generated from the Bäcklund transformation for the Tota lattice[J]. Suppl Prog Theor Phys,1976, 59: 64-100.

[4] Liu H Z. Exact periodic wave solutions for the hKdV equation[J]. Nonlinear Anal, 2009, 70: 2379-2381.

[5] Olver P J. Applications of Lie Groups to Differential Equations, in: Grauate Texts in Mathematics[M]. New York: Springer, 1993.

[6] Bluman G W, Kumei S. Symmetries and Differential Equations[M]. Berlin: Springer-Verlag World Publishing Gorp, 1989.

[7] Cantwell B J. Introduction to Symmetry Analysis[M]. Cambridge: Cambridge University Press, 2002.

[8] Liu H Z. Lie symmetry analysis and exact explicit solutions for general Burgers' equation[J]. J Comput Appl Math, 2008, 228: 1-9.

[9] Liu Y, Liu X Q. Symmetry reduction, exact solutions and conservation laws of the (1+1)dimension Caudrey-Dodd-Gibbon equation[J].

Journal of Liaocheng University (Natural Science Edition),2014,27(01):8-12+24.
[10] Li Z Q,Li u H Z. The application of (G′/G)-expansion method for fifth-order dispersive equation [J]. Journal of Liaocheng University (Natural Science Edition),2016,29(04):1-3+7.
[11] Li Y, Li L Q, Li H H,et al. Exact solutions and conservation laws of the (3+1)dimensional Zakharov-Kuznetsov-Burgers equation [J]. Journal of Liaocheng University(Natural Science Edition),2017,30(01):10-17.
[12] Xin X P. Non-local symmetries and exact solutions of nonlinear development equations[J]. Journal of Liaocheng University(Natural Science Edition),2018,31(1):15-20.
[13] Gesztesy F,Holden H. Soliton Equations and their Algebro-Geometric Solutions[M]. Cambridge: Cambridge University Press, 2003.
[14] Hou Y,Fan E G. Algebro-geometric solutions for the Gerdjikov-Ivanov hierarchy[J]. J Math Phys, 2013, 54: 073505-073530.
[15] Hou Y, Fan E G, Zhao P. The algebro-geometric solutions for Hunter-Saxton hierarchy[J]. Z Angew Math Phys,2013,65(3):487-520.
[16] Hou Y, Zhao P, Fan E G,et al. Algebro-geometric solutions for the Degasperis-Procesi hierarchy[J]. SIAM J Math Anal, 2013, 45: 1216-1266.
[17] Zhao P, Fan E G,Hou Y. Algebro-Geometric solutions for the Ruijsenaars-Toda hierarchy[J]. Chaos Solitons Fractals, 2013, 54: 8-25.
[18] Tu G Z. The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems[J]. J Math Phys, 1989, 30: 330-338.
[19] Zhang Y F,Hon Y C. Some evolution hierarchies derived from self-dual Yang-Mills equations[J]. Commun Theor Phys, 2011, 56: 856-872.
[20] Zhang Y F,Feng B L. A few Lie algebras and their applications for generating integrable hierarchies of evolution types[J]. Commun Nonlinear Sci Numer Simulat,2011, 16(8): 3045-3061.
[21] Zhang Y F, Tam H H. An integrable hierarchy and Darboux transformations, bilinear Bäcklund transformations of a reduced equation [J]. Applied Mathematics and Computation, 2013, 219:5837-5848.
[22] Wang X Z, Dong H H, Li Y X. Some reductions from a lax integrable system and their hamiltonian structures[J]. Applied Mathematics and Computation,2012, 218:10032-10039.
[23] Yang H X, Du J, Xu X X,et al. Hamiltonian and Super-hamiltonian systems of a hierarchy of soliton equations[J]. Applied Mathematics and Computation, 2010, 217: 1497-1508.

# 非线性 Schrodinger-MKdV 方程的 Hamilton 结构及代数几何解

岳 超

(泰山医学院 医学信息工程学院,山东 泰安 271016)

**摘 要** 由 $3\times3$ 等谱 Lax 矩阵导出了非线性 Schrödinger-MKdV(NLS-MKdV)方程族,应用迹恒等式得到了其 Hamilton 结构. 为方便构造代数几何解,我们将 $3\times3$ 矩阵等谱问题转化为等价的 $2\times2$ 问题,借助 Riemann theta 函数,求出了耦合的 NLS 方程及耦合的 MKdV 方程的代数几何解.

**关键词** 迹恒等式;Hamilton 结构;代数几何解;Riemann theta 函数