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Global Attractors for Non-densely Defined Evolution Equations

YOU Hong-lian

(School of Science, Binzhou University, Binzhou 256600, China)

Abstract In this paper, we consider the existence of global attractors for a class of evolution equations with or without delay, where the operator in the linear part is not necessarily densely defined. The novelty of this work is that we do not need the assumption in our previous works [6, 18] that the C_0-semigroup associated with the linear part is compact. Therefore, the method in this paper is applicable to a wide class of evolution equations. Techniques in this paper is the full use of a generalized Gronwall inequality and the Kuratowski's measure of non-compactness. Examples of the C_0-semigroup associated with the linear part being not compact is given as an application to illustrate the generalization of our result.

Key words global attractor; Hille-Yosida operator; measure of non-compactness; κ-contrating

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0 Introduction

In this paper, we study the existence of global attractors for the following evolution equation without delay

$$\begin{cases} u'(t)=Au(t)+F(u(t)),t>0, \\ u(0)=x_0\in X, \end{cases} \tag{1}$$

where $(X, \|\cdot\|)$ is a Banach space; and equation with finite delay

$$\begin{cases} u'(t)=Au(t)+F(u_t),t>0, \\ u(0)=\varphi\in C, \end{cases} \tag{2}$$

where $C=C([-r,0],X)$ is the space of continuous functions from $[-r,0]$ to X. For infinite delay case, it can be solved similarly, except that the phase space will be changed to others. A is a Hille-Yosida operator but not necessarily densely defined, which generates an integrated semigroup. F is a nonlinear functional determined later. As usual, for every $t\in\mathbf{R}$, the history function $u_t\in C$ is defined by

$$u_t(\theta)=u(t+\theta),\theta\in[-r,0].$$

On the study of dynamic behaviors of Eq. (1) and Eq. (2), we refer to works [1-3,5, 9, 16, 18] and the references listed therein for more information. Most of the papers mentioned above are devoted to the existence, regularity and stability of equilibria or solutions. Only the work in [5] and our previous work in [6] studied the existence of global attractors for Eq. (2) with finite delay, and meanwhile, the case with finite delay and without delay were considered in our another previous paper [18]. Comparatively speaking, works on specific semilinear PDEs other than the general form as Eq. (1) and Eq. (2) is a little more, see e. g. [4,7,14].

We note that the above mentioned works require that the semigroup associated with the linear part is compact. This condition is needed in previous works to guarantee the compactness of the solution operator, which is a necessary condition for the existence of the global attractor, see [10] for details. However, there are many examples in which the linear operator corresponds to a non-compact semigroup, e. g. , the

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通讯作者:由红连,女,汉族,博士,副教授,研究方向:无穷维动力系统,E-mail:hlyou_bz@163.com.

age-structured population model (see section 4). In recent paper [11], Eq. (1) with random perturbation is considered. Under some appropriate assumptions, they studied the existence, uniqueness and the long time behavior of solutions, as well as the existence of a global (random) attractor at last. In these assumptions, the linear part A generates an analytic semigroup, no compactness being needed. But another more condition that $(\beta_0 I-A)^{-1}$ is compact for some constant β_0 is needed to obtain the compactness of solutions. Therefore, it will make sense if the compactness assumption is removed, which is a difficulty and which also motivates this paper.

Except the compactness, the uniform estimation of solutions is another necessary condition for the existence of the global attractor. In the delay case and no-delay case, however, the methods of estimation are very different. In detail, in the delay case, we need to rely on a weighted norm to get the uniform estimation of solutions (see Proposition 5).

Therefore, the purpose of this paper is to remove the compactness assumption completely on the semigroup associated with the linear part, for both the cases with and without delay. To this end, under the basic theory of C_0-semigroup in [10] and integrated semigroup in [15], we will take full advantage of the Kuratowski's measure of non-compactness instead of Ascoli-Arzelà method to overcome the difficulty due to the absence of the compactness assumption, which is a novelty of this paper.

Compared with work in [11], another novelty is that the linear operator A is not necessarily densely defined. In fact, operators with non-dense domain occur in many situations owing to restrictions on the space where the equations are considered. For example, periodic continuous functions and Hölder continuous functions are not dense in the space of continuous functions; see more examples in [13].

In the next section, we list some preliminaries for our use. Then in section 3, we develop the existence of the global attractor for Eq. (1). Regarding results for Eq. (2) with finite delay, we study it in section 4. In the last section, we give some example to illustrate our results.

1 Preliminaries

In this section, we list some results on integrated semigroup and C_0-semigroup for our use, see [10, 15]. Let E be a general Banach space. Consider the following nonhomogeneous Cauchy problem

$$\begin{cases} u'(t)=Au(t)+f(t), t>0, \\ u(0)=x\in E, \end{cases} \tag{3}$$

where $f:[0,+\infty)\to E$ is continuous and $A:D(A)\subset E\to E$ satisfies the following assumption

(A) A is a closed linear operator, and there exist constants $\omega\in\mathbf{R}$ and $M\geqslant 1$ such that $(\omega,+\infty)\subset\rho(A)$ and

$$|(\lambda I-A)^{-n}|\leqslant\frac{M}{(\lambda-\omega)^n}, \lambda>\omega, \tag{4}$$

where $\rho(A)$ denotes the resolvent set of A and $|\cdot|$ is the operator norm.

Rmark 1 In the sequel of this paper, we always assume that the operator A satisfies assumption (A). Meanwhile, for simplicity, we take $M=1$ in (4). In fact, this can be done if we employ the renorming lemma in [12] to introduce an equivalent norm in E.

Definition 1 A continuous function $x:[0,+\infty)\to E$ is called an integral solution of Eq. (3) if $\int_0^t x(s)\mathrm{d}s\in D(A)$ and

$$x(t)=x+A\int_0^t x(s)\mathrm{d}s+\int_0^t f(s)\mathrm{d}s. \tag{5}$$

In order to study the non-densely defined operator, we introduce the part A_0 of A in $\overline{D(A)}$

$$A_0=A \text{ on } D(A_0)=\{x\in D(A):Ax\in\overline{D(A)}\}.$$

Proposition 1 [15]　The part A_0 of A in $\overline{D(A)}$ generates a C_0-semigroup $T_0(t)$, $t\geqslant 0$, on $\overline{D(A)}$.

Proposition 2 [15]　The unique continuous solution to Eq. (5) with values in $\overline{D(A)}$ are given by

$$u(t)=T_0(t)x+\lim_{\lambda\to+\infty}\int_0^t T_0(t-s)\lambda(\lambda I-A)^{-1}f(s)\,\mathrm{d}s.$$

The Kuratowski's measure of non-compactness for a bounded set B of a Banach space E is defined as

$$\kappa(B)=\inf\{d>0: B \text{ has a finite cover of diameter} <d\}.$$

Definition 2 [10]　A semigroup $T(t)$, $t\geqslant 0$, is said to be κ-contracting if there exists a continuous function $\beta: R^+\to R^+$ such that $\beta(t)\to 0$, as $t\to+\infty$, and for each $t>0$ and each bounded set $B\subset E$ with $\kappa(B)>0$, we have $\kappa(T(t)B)\leqslant\beta(t)\kappa(B)$.

Theorem 1 [10]　If $T(t): E\to E$, $t\geqslant 0$, is κ-contracting, point dissipative and orbits of bounded sets are bounded, then there exists a global attractor.

At the end of this section, we state a generalized Gronwall inequality in [8], which is essential to our proofs through this paper.

Lemma 1 [8]　If $x(t)\leqslant h(t)+\int_{t_0}^t g(s)x(s)\,\mathrm{d}s$, $t\in[t_0,T)$, where all the functions involved are continuous on $[t_0,T)$, $T\leqslant+\infty$, and $g(t)\geqslant 0$, then $x(t)$ satisfies

$$x(t)\leqslant h(t)+\int_{t_0}^t h(s)g(s)\mathrm{e}^{\int_s^t g(u)\mathrm{d}u}\,\mathrm{d}s,\ t\in[t_0,T).$$

2　Equations without delay

In this section, we consider Eq. (1). Assume that F satisfies the following assumption

(F_1)　$F: X\to X$ is Lipschitz continuous in the sense that there exists a constant $L_1>0$ such that

$$\|F(x)-F(y)\|\leqslant L_1\|x-y\| \text{ for all } x,y\in X.$$

Then for Eq. (1), it is well known that (A) and (F_1) are sufficient conditions to guarantee that its integral solution exists uniquely and globally. On the other hand, A is not necessarily densely defined. Let $X_0=\overline{D(A)}$. With the help of Propositions 1 and 2, the integral solution of Eq. (1) can be expressed as following

$$u(t)=T_0(t)x_0+\lim_{\lambda\to+\infty}\int_0^t T_0(t-s)\lambda(\lambda I-A)^{-1}F(u(s))\,\mathrm{d}s \text{ for all } t\geqslant 0, \tag{6}$$

where $T_0(t)$, $t\geqslant 0$, is the C_0-semigroup generated by A_0, the part of A in X_0. Moreover, for any integral solution u of Eq. (1), $u(t)\in\overline{D(A)}$. Then, for any $t\geqslant 0$, we can define $U(t): X_0\to X_0$ as following $U(t)x_0=u(t)$, where u is the integral solution of Eq. (1) initiated with x_0. Furthermore, $U(t)$, $t\geqslant 0$, is a C_0-semigroup on X_0, see [5].

In the following, we investigate the properties of $U(t)$. To this end, we make the following additional assumption on $T_0(t)$, $t\geqslant 0$.

(T)　$T_0(t)\leqslant\mathrm{e}^{-\alpha t}$, $\forall t\geqslant 0$, where $\alpha>0$ is some constant.

Proposition 3　Suppose that (A), (F_1) and (T) hold. Then, for any $x_0\in X_0$,

$$\|u(t)\|\leqslant\frac{\|F(0)\|}{\alpha-L_1}+\left(\|x_0\|-\frac{\|F(0)\|}{\alpha-L_1}\right)\mathrm{e}^{-(\alpha-L_1)t},\ t\geqslant 0, \tag{7}$$

where u is the integral solution of Eq. (1) initiated with x_0, $\alpha\neq L_1$.

Proof　From (6), for any $x_0\in X_0$, we have

$$\begin{aligned}\|u(t)\|&\leqslant\|T_0(t)x_0\|+\lim_{\lambda\to+\infty}\int_0^t\|T_0(t-s)\lambda(\lambda I-A)^{-1}F(u(s))\|\,\mathrm{d}s\\&\leqslant\mathrm{e}^{-\alpha t}\|x_0\|+\lim_{\lambda\to+\infty}\int_0^t\mathrm{e}^{-\alpha(t-s)}|\lambda(\lambda I-A)^{-1}|(\|F(0)\|+L_1\|u(s)\|)\,\mathrm{d}s\\&=\mathrm{e}^{-\alpha t}\|x_0\|+\|F(0)\|\mathrm{e}^{-\alpha t}\int_0^t\mathrm{e}^{\alpha s}\,\mathrm{d}s+L_1\mathrm{e}^{-\alpha t}\int_0^t\mathrm{e}^{\alpha s}\|u(s)\|\,\mathrm{d}s.\end{aligned}$$

Then

$$\mathrm{e}^{\alpha t}\|u(t)\|\leqslant\|x_0\|+\frac{\|F(0)\|}{\alpha}(\mathrm{e}^{\alpha t}-1)+L_1\int_0^t\mathrm{e}^{\alpha s}\|u(s)\|\,\mathrm{d}s.$$

By the generalized Gronwall inequality in Lemma 1, we get that

$$e^{\alpha t}\|u(t)\| \leqslant \|x_0\| + \frac{\|F(0)\|}{\alpha}(e^{\alpha t}-1) + L_1\int_0^t \left(\|x_0\| + \frac{\|F(0)\|}{\alpha}(e^{\alpha s}-1)\right)e^{L_1(t-s)}ds$$

$$\leqslant e^{L_1 t}\|x_0\| + \frac{\|F(0)\|}{\alpha - L_1}(e^{\alpha t} - e^{L_1 t}),$$

which implies that

$$\|u(t)\| \leqslant \frac{\|F(0)\|}{\alpha - L_1} + \left(\|x_0\| - \frac{\|F(0)\|}{\alpha - L_1}\right)e^{-(\alpha - L_1)t}, t \geqslant 0.$$

Lemma 2 Assume that the hypotheses in Proposition 3 hold. In addition, $\alpha > L_1$. Then $U(t)$, $t \geqslant 0$, is point dissipative and orbits of bounded sets are bounded.

Proof From Proposition 3, we know that, for any $x_0 \in X_0$, since $\alpha > L_1 > 0$, there exists $t_0 = t_0(x_0) > 0$ such that for $t \geqslant t_0$,

$$\|u(t)\| \leqslant \frac{\|F(0)\|}{\alpha - L_1} + 1. \text{ (independent of } x_0).$$

Therefore, $B_{X_0}\left(0, \frac{\|F(0)\|}{\alpha - L_1} + 1\right) \cap X_0 \subset X_0$ attracts each point of X_0, where $B_{X_0}\left(0, \frac{\|F(0)\|}{\alpha - L_1} + 1\right)$ denotes the open ball in X_0 with center 0 and radius $\frac{\|F(0)\|}{\alpha - L_1} + 1$. On the other hand, since $\alpha > L_1 > 0$, by (7) we have, for any $t \geqslant 0$,

$$\|u(t)\| \leqslant \frac{\|F(0)\|}{\alpha - L_1} + \|x_0\|,$$

which implies that orbits of bounded sets are bounded.

Proposition 4 Let (A), (F_1) and (T) hold. Then, for any $x_0, y_0 \in X_0$,

$$\|U(t)x_0 - U(t)y_0\| \leqslant e^{-(\alpha - L_1)t}\|x_0 - y_0\|, \forall t \geqslant 0. \tag{8}$$

Proof By (6), for any $x_0, y_0 \in X_0$, we have

$$\begin{aligned}
&\|U(t)x_0 - U(t)y_0\| \\
&\leqslant \|T_0(t)x_0 - T_0(t)y_0\| + \lim_{\lambda \to +\infty}\int_0^t \|T_0(t-s)\lambda(\lambda I - A)^{-1}(F(U(s)x_0) - F(U(s)y_0))\| ds \\
&\leqslant e^{-\alpha t}\|x_0 - y_0\| + \lim_{\lambda \to +\infty} L_1\int_0^t e^{-\alpha(t-s)}|\lambda(\lambda I - A)^{-1}|\,\|U(s)x_0 - U(s)y_0\| ds \\
&\leqslant e^{-\alpha t}\|x_0 - y_0\| + L_1 e^{-\alpha t}\int_0^t e^{\alpha s}\|U(s)x_0 - U(s)y_0\| ds.
\end{aligned}$$

Then

$$e^{\alpha t}\|U(t)x_0 - U(t)y_0\| \leqslant \|x_0 - y_0\| + L_1\int_0^t e^{\alpha s}\|U(s)x_0 - U(s)y_0\| ds.$$

Using the Gronwall inequality, we get that

$$e^{\alpha t}\|U(t)x_0 - U(t)y_0\| \leqslant e^{L_1 t}\|x_0 - y_0\|,$$

which implies that

$$\|U(t)x_0 - U(t)y_0\| \leqslant e^{-(\alpha - L_1)t}\|x_0 - y_0\|, \forall t \geqslant 0.$$

Lemma 3 Suppose (A), (F_1) and (T) hold true. Then $U(t): X_0 \to X_0$, $t \geqslant 0$, is κ-contracting provided that $\alpha > L_1$.

Proof Let $B \subset X_0$ be any bounded set with $\kappa(B) > 0$. Then for any $t \geqslant 0$, $U(t): B \to U(t)B$is surjective. Moreover, from (8) we know that

$$\|U(t)x_0 - U(t)y_0\| \leqslant e^{-(\alpha - L_1)t}\|x_0 - y_0\|, \ \forall x_0, y_0 \in B.$$

For any $\varepsilon > 0$, there exist bounded sets $B_i \subset B$, $i = 1, 2, \cdots, m$, such that $B = \bigcup_{i=1}^m B_i$ and

$$\operatorname{diam}\{B_i\} \leqslant \kappa(B) + \varepsilon, \ i = 1, 2, \cdots, m.$$

Since $U(t)B = \bigcup_{i=1}^m U(t)B_i$, for any $x, y \in U(t)B_i$, we assume that $x = U(t)x_0$, $y = U(t)y_0$ for some $x_0, y_0 \in B_i$. Thus

$$\| x-y \| = \| U(t)x_0 - U(t)y_0 \| \leqslant e^{-(\alpha-L_1)t} \| x_0 - y_0 \| \leqslant e^{-(\alpha-L_1)t} \mathrm{diam}(B_i) \leqslant e^{-(\alpha-L_1)t}(\kappa(B)+\varepsilon).$$

This implies that $\kappa(U(t)B_i) \leqslant e^{-(\alpha-L_1)t}(\kappa(B)+\varepsilon)$, and hence $\kappa(U(t)B) \leqslant e^{-(\alpha-L_1)t}(\kappa(B)+\varepsilon)$.

By the arbitrary of ε, we obtain that $\kappa(U(t)B) \leqslant e^{-(\alpha-L_1)t}\kappa(B)$.

Remark 2 In this lemma, the Kuratowski′s measure of non-compactness is key to the removement of the compact assumption on $T_0(t)$, as well as the finite delay case (see Lemma 5). One can refer to [5,18] that $T_0(t)$ need to be compact because they use Ascoli-Arzelà method to prove the compactness of the solution operator $U(t)$.

As a result of Theorem 1, Lemma 2 and 3, we give the main result of this section.

Theorem 2 Let (A), (F_1) and (T) hold. If $\alpha > L_1$, then Eq. (1) has a global attractor in X_0.

3 Equations with finite delay

In this section, we consider Eq. (2), in which the phase space $C([-r,0],X)$ is equipped with the supreme norm denoted by $\|\cdot\|_C$. To this end, we assume that

(F_2) $F: C([-r,0],X) \to X$ is Lipschitz continuous in the sense that there exists a constant $L_2 > 0$ such that

$$\| F(\varphi_1) - F(\varphi_2) \| \leqslant L_2 \| \varphi_1 - \varphi_2 \|_C \text{ for all } \varphi_1, \varphi_2 \in C([-r,0],X).$$

Let

$$\Sigma_0 = \{\varphi \in C([-r,0],X) \mid \varphi(0) \in \overline{D(A)}\}.$$

A similar argument as that in the first part of section 3, we get that the integral solution of Eq. (2) can be expressed as following

$$u_t(\theta) = \begin{cases} T_0(t+\theta)\varphi(0) + \lim\limits_{\lambda\to+\infty} \int_0^t T_0(t+\theta-s)\lambda(\lambda I - A)^{-1} F(u_s)\,ds, -t \leqslant \theta \leqslant 0, \\ \varphi(t+\theta), -r \leqslant \theta \leqslant -t, \end{cases} \tag{9}$$

where $T_0(t)$, $t \geqslant 0$, is the C_0-semigroup generated by A_0 on Σ_0. Then for any $t \geqslant 0$, define $U(t): \Sigma_0 \to \Sigma_0$ as following $U(t)\varphi = u_t$, where u is the integral solution of Eq. (2) initiated with φ, and $U(t)$, $t \geqslant 0$, is a C_0-semigroup on Σ_0[5].

In order to study the behavior of $U(t)$, we suppose that $T_0(t)$ satisfies condition (T) in section 2.

Proposition 5 Suppose that (A), (F_2) and (T) hold. Then for any $\varphi \in \Sigma_0$,

$$\| u_t \|_C \leqslant \frac{e^{\alpha r} \| F(0) \|}{\alpha - L_2 e^{\alpha r}} + e^{\alpha r}\left(\| \varphi \|_C - \frac{\| F(0) \|}{\alpha - L_2 e^{\alpha r}} \right) e^{-(\alpha - L_2 e^{\alpha r})t}, t \geqslant 0, \tag{10}$$

where u is the integral solution of Eq. (2) initiated with φ, $\alpha \neq L_2 e^{\alpha r}$.

Proof From (9), we prove this result in two cases.

Case 1. $0 \leqslant t \leqslant r$. Then

$$\begin{aligned}
&\sup_{-r\leqslant\theta\leqslant 0} e^{\alpha\theta} \| u_t(\theta) \| \\
&= \max\{ \sup_{-r\leqslant\theta\leqslant -t} e^{\alpha\theta} \| \varphi(t+\theta) \|, \sup_{-t\leqslant\theta\leqslant 0} e^{\alpha\theta} \| u_t(\theta) \| \} \\
&\leqslant \max\{ e^{-\alpha t} \| \varphi \|_C, \sup_{-t\leqslant\theta\leqslant 0} e^{\alpha\theta} e^{-\alpha(t+\theta)} \| \varphi(0) \| + \sup_{-t\leqslant\theta\leqslant 0} e^{\alpha\theta} \lim_{\lambda\to+\infty} \int_0^{t+\theta} e^{-\alpha(t+\theta-\tau)} |\lambda(\lambda I - A)^{-1}| (\| F(0) \| + L_2 \| u_\tau \|_C) d\tau \} \\
&\leqslant \max\{ e^{-\alpha t} \| \varphi \|_C, e^{-\alpha t} \| \varphi(0) \| + \sup_{-t\leqslant\theta\leqslant 0} e^{\alpha\theta} \int_0^{t+\theta} e^{-\alpha(t+\theta-\tau)} (\| F(0) \| + L_2 \| u_\tau \|_C) d\tau \} \\
&\leqslant e^{-\alpha t} \| \varphi \|_C + \| F(0) \| e^{-\alpha t} \int_0^t e^{\alpha\tau} d\tau + L_2 e^{-\alpha t} \int_0^t e^{\alpha\tau} \| u_\tau \|_C d\tau \\
&= e^{-\alpha t} \| \varphi \|_C + \frac{\| F(0) \|}{\alpha}(1 - e^{-\alpha t}) + L_2 e^{-\alpha t} \int_0^t e^{\alpha\tau} \| u_\tau \|_C d\tau.
\end{aligned}$$

On the other hand, we have

$$\sup_{-r\leqslant\theta\leqslant 0} e^{\alpha\theta} \| u_t(\theta) \| \geqslant e^{-\alpha r} \sup_{-r\leqslant\theta\leqslant 0} \| u_t(\theta) \| = e^{-\alpha r} \| u_t \|_C.$$

Then

$$e^{-\alpha r}\|u_t\|_C\leqslant e^{-\alpha t}\|\varphi\|_C+\frac{\|F(0)\|}{\alpha}(1-e^{-\alpha t})+L_2e^{-\alpha t}\int_0^t e^{\alpha\tau}\|u_\tau\|_C d\tau.$$

So we get

$$e^{\alpha t}\|u_t\|_C\leqslant e^{\alpha r}(\|\varphi\|_C+\frac{\|F(0)\|}{\alpha}(e^{\alpha t}-1))+L_2e^{\alpha r}\int_0^t e^{\alpha\tau}\|u_\tau\|_C d\tau.$$

By the generalized Gronwall inequality in Lemma 1, we obtain that

$$e^{\alpha t}\|u_t\|_C\leqslant e^{\alpha r}(\|\varphi\|_C+\frac{\|F(0)\|}{\alpha}(e^{\alpha t}-1))+L_2e^{\alpha r}\int_0^t e^{\alpha r}(\|\varphi\|_C+\frac{\|F(0)\|}{\alpha}(e^{\alpha s}-1))e^{\int_s^t L_2e^{\alpha r}d\tau}ds$$

$$=\frac{\|F(0)\|e^{\alpha r}}{\alpha-L_2e^{\alpha r}}e^{\alpha t}+e^{\alpha r}(\|\varphi\|_C-\frac{\|F(0)\|}{\alpha-L_2e^{\alpha r}})e^{L_2e^{\alpha r}t}.$$

Consequently, we get

$$\|u_t\|_C\leqslant\frac{\|F(0)\|e^{\alpha r}}{\alpha-L_2e^{\alpha r}}+e^{\alpha r}(\|\varphi\|_C-\frac{\|F(0)\|}{\alpha-L_2e^{\alpha r}})e^{-(\alpha-L_2e^{\alpha r})t},0\leqslant t\leqslant r.$$

Case 2. $t>r$. In this case, the integral solution does not include the initial part. Thus,

$$\sup_{-r\leqslant\theta\leqslant 0}e^{\alpha\theta}\|u_t(\theta)\|$$

$$\leqslant\sup_{0\leqslant t+\theta\leqslant t}e^{\alpha\theta}e^{-\alpha(t+\theta)}\|\varphi(0)\|+\sup_{0\leqslant t+\theta\leqslant t}e^{\alpha\theta}\lim_{\lambda\to+\infty}\int_0^{t+\theta}e^{-\alpha(t+\theta-\tau)}|\lambda(\lambda I-A)^{-1}|(\|F(0)\|+L_2\|u_\tau\|_C)d\tau$$

$$\leqslant e^{-\alpha t}\|\varphi(0)\|+\sup_{0\leqslant t+\theta\leqslant t}e^{\alpha\theta}\int_0^{t+\theta}e^{-\alpha(t+\theta-\tau)}(\|F(0)\|+L_2\|u_\tau\|_C)d\tau$$

$$\leqslant e^{-\alpha t}\|\varphi\|_C+\frac{\|F(0)\|}{\alpha}(1-e^{-\alpha t})+L_2e^{-\alpha t}\int_0^t e^{\alpha\tau}\|u_\tau\|_C d\tau.$$

Similarly as that in case 1, we get that

$$\|u_t\|_C\leqslant\frac{\|F(0)\|e^{\alpha r}}{\alpha-L_2e^{\alpha r}}+e^{\alpha r}\left(\|\varphi\|_C-\frac{\|F(0)\|}{\alpha-L_2e^{\alpha r}}\right)e^{-(\alpha-L_2e^{\alpha r})t},t>r.$$

Lemma 4 Assume that the hypotheses in Proposition 5 hold. In addition, $\alpha>L_2e^{\alpha r}$. Then $U(t)$, $t\geqslant 0$, is point dissipative and orbits of bounded sets are bounded.

Proof One can refer to Lemma 2 for a similar proof.

Proposition 6 Let (A), (F_2) and (T) hold. Then for any $\varphi,\psi\in\Sigma_0$,

$$\|U(t)\varphi-U(t)\psi\|_C\leqslant e^{\alpha r}e^{-(\alpha-L_2e^{\alpha r})t}\|\varphi-\psi\|_C,\forall t\geqslant 0.$$

Proof For any $\varphi,\psi\in\Sigma_0$, from (9), we prove this proposition in two cases.

Case 1 $0\leqslant t\leqslant r$. Then

$$\sup_{\theta\in[-r,0]}e^{\alpha\theta}\|(U(t)\varphi)(\theta)-(U(t)\psi)(\theta)\|$$

$$\leqslant\max\{\sup_{\theta\in[-r,-t]}e^{\alpha\theta}\|\varphi(t+\theta)-\psi(t+\theta)\|,\sup_{\theta\in[-t,0]}e^{\alpha\theta}\|T_0(t+\theta)(\varphi(0)-\psi(0))\|$$

$$+\sup_{\theta\in[-t,0]}e^{\alpha\theta}\lim_{\lambda\to+\infty}\int_0^{t+\theta}\|T_0(t+\theta-s)\lambda(\lambda I-A)^{-1}(F(U(s)\varphi)-F(U(s)\psi))\|ds\}$$

$$\leqslant\max\{e^{-\alpha t}\|\varphi-\psi\|_C,\sup_{\theta\in[-t,0]}e^{\alpha\theta}e^{-\alpha(t+\theta)}\|\varphi(0)-\psi(0)\|$$

$$+L_2\sup_{\theta\in[-t,0]}e^{\alpha\theta}\lim_{\lambda\to+\infty}\int_0^{t+\theta}e^{-\alpha(t+\theta-s)}|\lambda(\lambda I-A)^{-1}|\|U(s)\varphi-U(s)\psi\|_C ds\}$$

$$\leqslant\max\{e^{-\alpha t}\|\varphi-\psi\|_C,e^{-\alpha t}\|\varphi(0)-\psi(0)\|+L_2e^{-\alpha t}\int_0^t e^{\alpha s}\|U(s)\varphi-U(s)\psi\|_C ds\}$$

$$\leqslant e^{-\alpha t}\|\varphi-\psi\|_C+L_2e^{-\alpha t}\int_0^t e^{\alpha s}\|U(s)\varphi-U(s)\psi\|_C ds.$$

Since

$$\sup_{\theta\in[-r,0]}e^{\alpha\theta}(U(t)\varphi)(\theta)-(U(t)\psi)(\theta)\geqslant e^{-\alpha r}\|U(t)\varphi-U(t)\psi\|_C,$$

then

$$e^{\alpha t}\|U(t)\varphi-U(t)\psi\|_C\leqslant e^{\alpha r}\|\varphi-\psi\|_C+L_2e^{\alpha r}\int_0^t e^{\alpha s}\|U(s)\varphi-U(s)\psi\|_C ds.$$

By the Gronwall inequality, we get that

$$\|U(t)\varphi-U(t)\psi\|_C\leqslant e^{\alpha r}e^{-(\alpha-L_2e^{\alpha r})t}\|\varphi-\psi\|_C,0\leqslant t\leqslant r.$$

Case 2 $t>r$. In this case, the integral solution does not include the initial part. Thus,

$$\sup_{\theta\in[-r,0]} e^{\alpha\theta}\|(U(t)\varphi)(\theta)-(U(t)\psi)(\theta)\|$$
$$\leqslant \sup_{\theta\in[-r,0]} e^{\alpha\theta}\|T_0(t+\theta)(\varphi(0)-\psi(0))\|+\sup_{\theta\in[-r,0]} e^{\alpha\theta}\lim_{\lambda\to+\infty}\int_0^{t+\theta}\|T_0(t+\theta-s)\lambda(\lambda I-A)^{-1}(F(U(s)\varphi)-F(U(s)\psi))\|ds$$
$$\leqslant \sup_{\theta\in[-r,0]} e^{\alpha\theta}e^{-\alpha(t+\theta)}\|\varphi(0)-\psi(0)\|+L_2\sup_{\theta\in[-r,0]} e^{\alpha\theta}\lim_{\lambda\to+\infty}\int_0^{t+\theta}e^{-\alpha(t+\theta-s)}|\lambda(\lambda I-A)^{-1}|\,\|U(s)\varphi-U(s)\psi\|_C ds$$
$$\leqslant e^{-\alpha t}\|\varphi-\psi\|_C+L_2e^{-\alpha t}\int_0^t e^{\alpha s}\|U(s)\varphi-U(s)\psi\|_C ds.$$

Similarly as that in case 1, we obtain that

$$\|U(t)\varphi-U(t)\psi\|_C\leqslant e^{\alpha r}e^{-(\alpha-L_2e^{\alpha r})t}\|\varphi-\psi\|_C,t>r.$$

Lemma 5　Suppose that (A), (F_2) and (T) hold. Then $U(t)$ is κ-contracting provided that $\alpha>L_2e^{\alpha r}$.

Proof　Similarly as the proof in Lemma 3, let $B\subset\Sigma_0$ be any bounded set with $\kappa(B)>0$, we have

$$\kappa(U(t)B)\leqslant e^{\alpha r}e^{-(\alpha-L_2e^{\alpha r})t}\kappa(B),\ \forall t\geqslant 0.$$

Since $\alpha>L_2e^{\alpha r}$, then $e^{\alpha r}e^{-(\alpha-L_2e^{\alpha r})t}\to 0$ as $t\to+\infty$, i. e. , $U(t)$is κ-contracting.

Following from Theorem 1, Lemmas 4 and 5, we arrive at the main theorem of the finite delay case.

Theorem 3　Let (A), (F_2) and (T) hold. If $\alpha>L_2e^{\alpha r}$, then Eq. (2) has a global attractor in Σ_0.

4　Applications

Considering the age-structured population model [17], which is a non-densely defined and non-compact example.

$$\begin{cases}u_t+u_a=-\mu(a)u(t,a)+\int_0^{+\infty}\gamma(a,b,u(t,b))db,t>0,a>0,\\ u(t,0)=\beta\int_0^{+\infty}u(t,a)da,t>0,\\ u(0,a)=\varphi(a),a>0,\end{cases}\tag{11}$$

with $u(t,\cdot)\in L^1(0,+\infty)$, the space of Lebesgue integrable functions with values in **R** and with norm denoted by $\|\cdot\|_{L^1}$; $u(t,a)$ represents the density of individuals of age a and time t; $\mu(a)\geqslant 0$ is the natural mortality; $\beta\geqslant 0$ denotes the fertility; $\gamma:\mathbf{R}_+\times\mathbf{R}_+\times\mathbf{R}\to\mathbf{R}$ is associated with a density-dependent migration process into or out of the population $\mathbf{R}_+$ is the set of nonnegative real numbers), and $\int_0^{+\infty}\gamma(\cdot,b,\varphi(b))db\in L^1(0,+\infty)$ for any $\varphi\in L^1(0,+\infty)$.

Let $X=\mathbf{R}\times L^1(0,+\infty)$ with the usual product norm of $\mathbf{R}\times L^1(0,+\infty)$, and $X_+=\mathbf{R}_+\times L^1_+(0,+\infty)$, Here $\varphi\in L^1_+(0,+\infty)$ means $\varphi\in L^1(0,+\infty)$ and $\varphi(a)\in\mathbf{R}_+$ for any $a>0$. Define $A:D(A)\subset X\to X$ as following

$$A\begin{pmatrix}0\\ \varphi\end{pmatrix}=\begin{pmatrix}-\varphi(0)\\ -\varphi'-\mu\varphi\end{pmatrix},\begin{pmatrix}0\\ \varphi\end{pmatrix}\in D(A),\tag{12}$$

where

$$D(A)=\{0\}_{\mathbf{R}}\times\{\varphi\in L^1(0,+\infty):\varphi'\in L^1(0,+\infty),\varphi(0)=0\}.$$

Clearly, $X_0=\overline{D(A)}=\{0\}_{\mathbf{R}}\times L^1(0,+\infty)\neq X$, i. e, A is non-densely defined. Define the nonlinear term $F:X_0\to X$ as following

$$F\left(\begin{pmatrix}0\\ \varphi\end{pmatrix}\right)=\begin{pmatrix}\beta\int_0^{+\infty}\varphi(a)da\\ \int_0^{+\infty}\gamma(\cdot,b,\varphi(b))db\end{pmatrix}.$$

Set $v(t)=\begin{pmatrix}0\\ u(t,\cdot)\end{pmatrix}\in X_0$ and $\begin{pmatrix}0\\ \varphi\end{pmatrix}=x\in X_0$, then Eq. (11) can be written as

$$\begin{cases}v'(t)=Av(t)+F(v(t)),t>0,\\ v(0)=x\in X_0.\end{cases}\tag{13}$$

Proposition 7　Suppose that there exists a constant $\bar{\mu}>0$ such that

$$\mu(a)\geqslant\bar{\mu},\ \forall a>0.\tag{14}$$

Then

(i) A, defined in (12), is a Hille-Yosida operator with $(-\bar{\mu},+\infty)\subset\rho(A)$ and

$$|(\lambda I-A)^{-1}|\leqslant\frac{1}{\lambda+\bar{\mu}},\ \forall\lambda>-\bar{\mu};$$

(ii) $(\lambda I-A)^{-1}X_+\subset X_+$, $\forall\lambda>\bar{\mu}$;

(iii) the C_0-semigroup $T_0(t)$, generated by A_0 on X_0, is not compact, but satisfies that

$$T_0(t)\leqslant e^{-\bar{\mu}t},\ \forall t\geqslant 0.$$

Proof (i) From (12), we know that

$$(\lambda I-A)\begin{pmatrix}0\\ \varphi\end{pmatrix}=\begin{pmatrix}\varphi(0)\\ \varphi'+(\lambda+\mu)\varphi\end{pmatrix}.$$

Set $y=\varphi(0)$ and $\zeta=\varphi'+(\lambda+\mu)\varphi$. Then

$$\varphi(a)=e^{-\lambda a-\int_0^a\mu(s)ds}y+\int_0^a e^{-\lambda(a-s)-\int_s^a\mu(\tau-s)d\tau}\zeta(s)ds \tag{15}$$

and $\varphi\in L^1(0,+\infty)$ provided that $\lambda>-\bar{\mu}$. Therefore, for any $\lambda>-\bar{\mu}$,

$$(\lambda I-A)^{-1}\begin{pmatrix}y\\ \zeta\end{pmatrix}=\begin{pmatrix}0\\ \varphi\end{pmatrix},$$

if and only if (15) holds. A simple calculation shows that

$$|(\lambda I-A)^{-1}|\leqslant\frac{1}{\lambda+\bar{\mu}},\ \forall\lambda>-\bar{\mu}.$$

(ii) This assertion follows from (15) since $y\geqslant 0$, $\zeta\in L_+^1(0,+\infty)$ implies $\varphi(a)\geqslant 0$ for any $a>0$.

(iii) The C_0-semigroup $T_0(t)$, generated by A_0 on X_0, possesses the following form

$$T_0(t)\begin{pmatrix}0\\ \varphi\end{pmatrix}=\begin{pmatrix}0\\ \hat{T}_0(t)\varphi\end{pmatrix},\text{where } \hat{T}_0(t)\varphi=\begin{cases}0, a<t,\\ \varphi(a-t)e^{-\int_{a-t}^a\mu(\tau)d\tau}, a\geqslant t.\end{cases}$$

Then $T_0(t)$ is not compact and

$$\left\|T_0(t)\begin{pmatrix}0\\ \varphi\end{pmatrix}\right\|=\|\hat{T}_0(t)\varphi\|_{L^1}=\int_t^{+\infty}|\varphi(a-t)e^{-\int_{a-t}^a\mu(\tau)d\tau}|da$$
$$=\int_0^{+\infty}|\varphi(a)e^{-\int_a^{a+t}\mu(\tau)d\tau}|da\leqslant e^{-\bar{\mu}t}\|\varphi\|_{L^1},$$

which implies that $T_0(t)\leqslant e^{-\bar{\mu}t}$, $\forall t\geqslant 0$.

We need the following assumptions on γ.

(H_γ) There exists a nonnegative function $L(\cdot)\in L^1(0,+\infty)$ such that

$$\int_0^{+\infty}|\gamma(a,b,\varphi_1(b))-\gamma(a,b,\varphi_2(b))|db\leqslant L(a)\|\varphi_1-\varphi_2\|_{L^1},\ \forall a>0.$$

and for any $r>0$, there exists a $c(r)>0$ such that

$$\int_0^{+\infty}\gamma(\cdot,b,\varphi(b))db+c(r)\varphi\in L_+^1(0,+\infty),\varphi\in L_+^1(0,+\infty),\ \|\varphi\|_{L^1}\leqslant r.$$

Under the assumptions of (H_γ), Eq. (13) possesses a unique global integral solution v. Then define $U(t):X_0\to X_0$ as $U(t)x=v(t)$, $\forall t\geqslant 0$.

Remark 3 From Proposition 5.1[17] (ii) and (H_γ), it is easy to prove that, for any $x\in X_{0+}$, the integral solution $v(t)\in X_{0+}$, $\forall t\geqslant 0$, i.e, X_{0+} is invariant with respect to $U(t)$. Therefore, we apply Theorem 2 to Eq. (13) in X_{0+} and arrive at the following result.

Theorem 4 Suppose that (H_γ) and (14) hold true. If $\bar{\mu}>\beta+\|L\|_{L^1}$, then Eq. (11) has a global attractor in $L_+^1(0,+\infty)$.

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非稠定发展方程的全局吸引子

由红连

(滨州学院 理学院,山东 滨州 256600)

摘　要　本文考虑一类发展方程在时滞存在和不存在两种情况下其全局吸引子的存在性,其中方程的线性部分不要求稠定.与前期工作[6,18]不同,此处去掉了线性算子生成的 C_0-半群的紧性假设,因此,本文中的方法适用范围更广.采用的主要技巧是广义的 Gronwall 不等式和 Kuratowski 非紧测度.作为对文中结论的应用,给出了线性算子生成的 C_0-半群非紧的例子.

关键词　全局吸引子;Hille-Yosida 算子;非紧测度;κ-压缩