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Invariant Tori for the Forth Order Nonlinear Schrödinger Equation with Unbounded Perturbation

CUI Wen-yan MI Lu-fang YOU Hong-lian

(School of Science, Binzhou University, Binzhou 256600, China)

Abstract In this paper, we are concerned with the forth order nonlinear Schrödinger (NLS) equation $iu_t+u_{xxxx}+|u_x|^2u_{xx}=0$, subject to Dirichlet boundary conditions. Using an infinite dimensional KAM theorem, we prove that there exist many n-dimensional invariant tori and thus many time quasi-periodic solutions for the above equation.

Key words Schrödinger equation; KAM theorem; quasi-periodic solution; normal form

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0 Introduction and main results

Consider an abstract evolution equation

$$\dot{w}=Aw+F(w), \tag{1}$$

where w is in some Hilbert space, say, Sobolev space H^p, and $A: H^p\rightarrow H^{p-d}$ is a linear operator, F is a nonlinear map sending some neighbourhood of H^p to $H^{p-\delta}$. $d\geqslant 1$ and $\delta\in\mathbf{R}$ are respectively defined as the orders of A and F. If $\delta\leqslant 0$, F is named a bounded perturbation, and if $\delta>0$, F is named an unbounded perturbation.

In general, we assume $\delta\geqslant d-1$ to guarantee the existence of KAM tori for the partial differential equations (PDEs), which based on Lax[5] and Klainerman[13] (see also [10]).

The existence of KAM tori of the PDEs with bounded Hamiltonian perturbations has been deeply and widely investigated by many researchers. The results in this field are too much to list here. We give just two survey papers [2,12]. Moreover, there are also some results of KAM theory for the PDEs with unbounded Hamiltonian perturbations. The earliest KAM theorem for unbounded perturbations is due to Kuksin[11] where it is supposed that $0<\delta<d-1$. In [11], Kuksin proved the persistence of the finite-gap solutions alongside the hierarchy of KdV equation with periodic boundary conditions. In 2010, Liu and Yuan[7] obtained a new estimate for the solution of the small-denominators equation with critical unbounded variable coefficients. With the new estimate, a KAM theorem for infinite dimensional Hamiltonian including $0<\delta<d-1$ and limiting case $\delta=d-1$ was established in [8].

The KAM theory not only for Hamiltonian systems but also for reversible systems has attracted great attention in the past few years. The KAM theory for the finite dimensional reversible system was put forward by Arnold[1], Moser[9] and investigated thoroughly by Seryuk[14-17], Broer[3] and Liu[a] mong others. Furthermore, Zhang-Gao-Yuan [18] initiated a KAM theorem for infinite dimensional reversible system with unbounded perturbation and obtained quasi-periodic solutions of a class of nonlinear Schrödinger equation with derivative in nonlinear terms, subject to Dirichlet boundary conditions.

Schrödinger equations are basic equations of quantum mechanics which were proposed to describe the motion of microscopic particles in 1926, after then it were widely used in the fields of atoms, molecules, solid physics, nuclear physics, and chemistry, and so on. There are plenty of results about the solutions of Schrödinger equations. For instance, In [27], He-Qian-Zou studied the existence and concentration of positive solutions for a class of quasilinear Schrödinger equations

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通讯作者:弭鲁芳,女,汉族,博士,教授,研究方向:Hamilton动力系统、不等式理论等,E-mail:milufang@126.com.

$$i\varepsilon\frac{\partial\psi}{\partial t}=-\varepsilon^2\Delta\psi+W(x)\psi-I(|\psi|^2)\psi-k\varepsilon^2\Delta[\rho(|\psi|^2)\psi]$$

with critical growth. More conclusions about the Schrödinger equations can be found in [23-26, 28] and the references therein.

Recently, many researchers focus on the existence of KAM tori and quasi-periodic solutions of Schrödinger equations. For example, in 2015, Feola-Procesi[4] considered a class of fully nonlinear forced and reversible Schrödinger equations

$$iu_t=u_{xx}+\varepsilon f(\omega t,x,u,u_x,u_{xx}),x\in T:=\mathbf{R}/2\pi\mathbf{Z},$$

and proved existence and stability of quasi-periodic solutions. In 2017, Lou-Geng[21] disscussed the existence of quasi-periodic response solutions for a class of reversible forced harmonic oscillators with two basic frequencies $\omega=(1,\alpha)$ where α is an irrational number. For more results about the KAM tori of Schrödinger equations, the readers can refer to [19, 20, 22].

However, there are few results on the existence of time quasi-periodic solutions for higher order reversible PDEs up to now. In the following we will focus on a forth order NLS equation

$$iu_t+u_{xxxx}+F(u,\bar{u},u_x,\bar{u}_x,u_{xx},\bar{u}_{xx})=0,(t,x)\in[0,\pi]\times\mathbf{R} \tag{2}$$

under the Dirichlet boundary conditions

$$u(t,0)=u(t,\pi)=0. \tag{3}$$

If Fsatisfies

$$F(\bar{u},u,\bar{u}_x,u_x,\bar{u}_{xx},u_{xx})=\overline{F(u,\bar{u},u_x,\bar{u}_x,u_{xx},\bar{u}_{xx})}$$

then (1) is reversible with respect to the involution $G:(u,\bar{u})\to(\bar{u},u)$. That is,

$$\boldsymbol{DG}\cdot\boldsymbol{X}=-\boldsymbol{X}\circ G,$$

where $\boldsymbol{DG}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $\boldsymbol{X}$ is the vector field of (2) with

$$\boldsymbol{X}=\begin{pmatrix}-i(u_{xxxx}+F(u,\bar{u},u_x,\bar{u}_x,u_{xx},\bar{u}_{xx}))\\ i(\bar{u}_{xxxx}+F(\bar{u},u,\bar{u}_x,u_x,\bar{u}_{xx},u_{xx}))\end{pmatrix}.$$

Let $F(u,\bar{u},u_x,\bar{u}_x,u_{xx},\bar{u}_{xx})=|u_x|^2u_{xx}$, we consider the following NLS equation

$$iu_t+u_{xxxx}+|u_x|^2u_{xx}=0,(t,x)\in[0,\pi]\times\mathbf{R} \tag{4}$$

subject to Dirichlet boundary conditions

$$u(t,0)=u(t,\pi)=0. \tag{5}$$

In this case, the equation (4) is reversible system with unbounded perturbation, but it is not Hamiltonian with symplectic structure $J=i$ (The proof of this conclusion is in the appendix of this paper).

In this paper, basing on the KAM theory for unbounded perturbation vector-field in [18], we can obtain the existence of many time-periodic solutions for equation (4). Our main work is to reduce the infinite dimensional coordinates form of (8) into a normal form up to order three and extract parameters. The plague we meet is to check the condition (20). Because of the complexity of higher order frequency, the matrixes of transformation $\boldsymbol{A},\boldsymbol{B}_1$ in (21) become very complex. Fortunately, we conquer it by careful computation and analysis.

In order to look for the solution u of equation (4), let

$$\varphi_j(x)=\sqrt{\frac{2}{\pi}}\sin jx,\lambda_j=j^4,j\geqslant1$$

be the basic modes and their frequencies for the linear equation $iu_t+u_{xxxx}=0$, with Dirichlet boundary conditions, then each solution is the superposition of oscillations of the basic modes, with the coefficients moving on circles $u(t,x)=\sum_{j\geqslant1}q_j(t)\varphi_j(x),q_j(t)=q_j^0\mathrm{e}^{i\lambda_jt}$.

Using the KAM theory in [18], we obtain the following result.

Theorem 1 Consider NLS equation (4) subject to Dirichlet boundary conditions (5). For all $n\in\mathbf{N}=\{1,2,\cdots\}$ and $J=\{j_1<j_2<\cdots<j_n\}$, there is an invariant linear space E_J of complex dimension n which is completely foliated into rotational tori

$$E_J=\{u=q_{j_1}\varphi_{j_1}+\cdots+q_{j_n}\varphi_{j_n}:q\in\mathbf{C}^n\}=\bigcup_{I\in P^n}T_J(I),$$

where $P^n=\{I:I_{j_i}=|q_{j_i}|^2>0\quad\text{for}\quad 1\leqslant i\leqslant n\}$ is the positive quadrant in $\mathbf{R}^n$ and

$$T_J(I)=\{u=q_{j_1}\varphi_{j_1}+\cdots+q_{j_n}\varphi_{j_n}:|q_{j_i}|^2=I_{j_i}\quad \text{for}\quad 1\leqslant i\leqslant n\}$$

such that there exists a positive measure cantor set $\Pi\subset P^n$ with positive measure and a family of n-dimensional tori $T_J(\Pi)=\bigcup_{I\in\Pi}T_J(I)\subset E_J$ given by a Lipschitz continuous embedding $\tilde{\varphi}:T_J(\Pi)\to H^p$ which is a higher order perturbation of the inclusion map $\tilde{\varphi}_0:E_J\to H^p$ restricted to $T_J(\Pi)$. Restriction of $\tilde{\varphi}$ to each $T_J(I)$ in the family is an embedding of a rotational n-dimensional torus of (4) and it carries quasi-periodic solution of (4).

1 Birkhoff normal form

At first, using a scalar transformation $u=\sqrt{\varepsilon}\tilde{u}$ (ε is a small positive real number), we change (4) into

$$i\tilde{u}_t+\tilde{u}_{xxxx}+\varepsilon|\tilde{u}_x|^2\tilde{u}_{xx}=0,\tag{6}$$

under Dirichlet boundary conditions $\tilde{u}(t,0)=\tilde{u}(t,\pi)=0$. In the following, we omit "$\sim$" for simplicity, still consider the equation

$$iu_t+u_{xxxx}+\varepsilon|u_x|^2u_{xx}=0,(t,x)\in[0,\pi]\times\mathbf{R}.\tag{7}$$

In order to write (7)in infinitely many coordinates, we let $u(t,x)=\sum_{j\geqslant1}q_j(t)\varphi_j(x)$ with $\varphi_j(x)=\sqrt{\frac{2}{\pi}}\sin jx$. The coordinates are taken from the Hilbert space ℓ^p of all complex-valued sequences $(q_j)_{j\geqslant1}$ with $\|\boldsymbol{q}\|_p^2=\sum_{j\geqslant1}|j|^{2p}|q_j|^2<\infty,p>\frac{5}{2}$.

Substitute it into (7), then (7) can be rewritten as

$$\dot{q}_j=ij^4q_j-i\varepsilon\sum_{l\pm s\pm m\pm j=0}lsm^2W_{lsmj}q_l\bar{q}_sq_m,\tag{8}$$

where $W_{lsmj}=\left(\frac{2}{\pi}\right)^2\int_0^\pi\cos lx\cos sx\sin mx\sin jx\mathrm{d}x$.

Meanwhile,it is easy to verify that (8) is reversible with respect to the involution $G_0:(q,\bar{q})\to(\bar{q},q)$.

To continue our investigation of the system (8), we need to establish the regularity of the nonlinear vector field

$$\boldsymbol{w}=(w_j)_{j\geqslant1}=\Big(\sum_{l\pm s\pm m\pm j=0}lsm^2W_{lsmj}q_l\bar{q}_sq_m\Big)_{j\geqslant1}.\tag{9}$$

Lemma 1 For $p>\frac{5}{2}$, we have

$$\|w\|_{p-2}=O(\|\boldsymbol{q}\|_p^3).\tag{10}$$

Proof Defining $\boldsymbol{r}=(r_j)_j=(jq_j)_j$, $\boldsymbol{r}'=(r'_j)_j=(j^2q_j)_j$, hence $|w_j|\leqslant C|(\boldsymbol{r}*\boldsymbol{r}*\boldsymbol{r}')_j|$.

For $\boldsymbol{q}\in\ell^p$, we have $\boldsymbol{r},\boldsymbol{r}'\in\ell^{p-2}$. Moreover, the space ℓ^{p-2} is a Banach algebra for $p>\frac{5}{2}$. So we have

$$\begin{aligned}\|\boldsymbol{w}\|_{p-2}&=\sqrt{\sum_{j\geqslant1}j^{2(p-2)}|w_j|^2}\leqslant C\sqrt{\sum_{j\geqslant1}j^{2(p-2)}|(\boldsymbol{r}*\boldsymbol{r}*\boldsymbol{r}')_j|^2}\\&=C\|\boldsymbol{r}*\boldsymbol{r}*\boldsymbol{r}'\|_{p-2}\leqslant C\|\boldsymbol{r}\|^2\|\boldsymbol{r}'\|_{p-2}\leqslant C\|\boldsymbol{q}\|^3.\end{aligned}$$

Lemma 2 If positive integers l,s,m,j satisfy $l\pm s\pm m\pm j=0$ and $\{l,m\}\neq\{s,j\}$, then we have

$$|l^4-s^4+m^4-j^4|\geqslant2jm.\tag{11}$$

Proof If $l+s+m-j=0$,we have $l+m=j-s$, thus $(l+m)^4=(j-s)^4$. It then follows that

$$\begin{aligned}l^4+m^4-s^4-j^4&=-4s^3j+6s^2j^2-4sj^3-4l^3m-6l^2m^2-4lm^3\\&=-sj(4(j-s)^2+2sj)-ml(4(m+l)^2-2ml)\\&=-4(j-s)^2(sj+ml)-2(s^2j^2-m^2l^2)\\&=-(sj+ml)(4(j-s)^2+2(sj-ml))\\&=-(sj+m(j-s-m))(s^2+j^2+m^2+l^2+(j-s)^2+(l+m)^2)\\&=(m+s)(m-j)(s^2+j^2+m^2+l^2+(j-s)^2+(l+m)^2),\end{aligned}$$

therefore, we obtain $|l^4-s^4+m^4-j^4|\geqslant j^2+m^2\geqslant2jm$.

By the same method, it is easy to know when $l\pm s\pm m\pm j=0$, we have

$$\begin{aligned}|l^4-s^4+m^4-j^4|&=|lm\pm sj\|(s^2+j^2+m^2+l^2+(s\pm j)^2+(l\pm m)^2)|\\&=|m\pm s\|m\pm j\|(s^2+j^2+m^2+l^2+(s\pm j)^2+(l\pm m)^2)|\\&\geqslant j^2+m^2\geqslant2mj.\end{aligned}$$

Lemma 3 There exists a transformation ψ which is bounded in a small neighborhood of the origin in

ℓ^p and change (8) into

$$\dot{z}_j = ij^4 z_j - i\varepsilon\sum_{l\geqslant 1} l^2 j^2 W_{lljj} |z_l|^2 z_j + i\varepsilon g_j(\boldsymbol{z},\bar{\boldsymbol{z}}), j\geqslant 1, \tag{12}$$

where $\| g(\boldsymbol{z},\bar{\boldsymbol{z}}) \|_{p-2} = O(\|\boldsymbol{q}\|_p^5)$ with $g(\boldsymbol{z},\bar{\boldsymbol{z}}) = (g_j(\boldsymbol{z},\bar{\boldsymbol{z}}))_{j\geqslant 1}$. Moreover, if (8) is reversible with respect to $G_0:(\boldsymbol{q},\bar{\boldsymbol{q}})\to(\bar{\boldsymbol{q}},\boldsymbol{q})$, then (12) is also reversible with respect to $G_0:(\boldsymbol{z},\bar{\boldsymbol{z}})\to(\bar{\boldsymbol{z}},\boldsymbol{z})$.

Proof We define a change ψ in variables

$$z_j = q_j + \varepsilon \sum_{l\pm s\pm m\pm j=0} T_{lsmj} q_l \bar{q}_s q_m, j\geqslant 1, \tag{13}$$

with coefficients

$$T_{lsmj} = \begin{cases} \dfrac{lsm^2 W_{lsmj}}{l^4 - s^4 + m^4 - j^4}, l\pm s\pm m\pm j=0, \{l,m\}\neq\{s,j\}, \\ 0, \text{otherwise}. \end{cases} \tag{14}$$

By Lemma 2, we see that (14) is well defined. Furthermore, we have

$$|T_{lsmj} q_l \bar{q}_s q_m| = \left| \frac{lsm^2 W_{lsmj} q_l \bar{q}_s q_m}{l^4 - s^4 + m^4 - j^4} \right| \leqslant \frac{|W_{lsmj}||lq_l||s\bar{q}_s||mq_m|}{2j}.$$

Denote $\tilde{\boldsymbol{q}} = (jq_j)_{j\geqslant 1}$. Then for $p > \frac{5}{2}$, using the algebra property of ℓ^{p-1}, we obtain

$$\| (\sum_{l\pm s\pm m\pm j=0} T_{lsmj} q_l \bar{q}_s q_m)_{j\geqslant 1} \|_p \leqslant \frac{1}{4\pi} \| \tilde{\boldsymbol{q}} * \tilde{\boldsymbol{q}} * \tilde{\boldsymbol{q}} \|_{p-1} \leqslant c(p) \| \tilde{\boldsymbol{q}} \|_{p-1}^3 \leqslant c(p) \| \boldsymbol{q} \|_p^3,$$

where the constant $c(p)$ depends only on p. Thus, the change in variables ψ is analytic in some neighborhood of the origin in ℓ^p into ℓ^p. It is easy to know that the change (13) is invertible and its inverse is of the form

$$q_j = z_j - \varepsilon \sum_{l\pm s\pm m\pm j=0} T_{lsmj} z_l \bar{z}_s z_m + O(\|\boldsymbol{z}\|_p^5), \quad j\geqslant 1 \tag{15}$$

in some neighborhood of the origin in ℓ^p. Differentiating both sides of (13) with respect to t, we have

$$\dot{z}_j = ij^4 q_j - i\varepsilon \sum_{l\pm s\pm m\pm j=0} lsm^2 W_{lsmj} q_l \bar{q}_s q_m + i\varepsilon \sum_{l\pm s\pm m\pm j=0} (l^4 - s^4 + m^4) T_{lsmj} q_l \bar{q}_s q_m + O(\|\boldsymbol{q}\|_p^5).$$

Substituting (15) into the last formula, we have

$$\dot{z}_j = ij^4 z_j - i\varepsilon \sum_{l\pm s\pm m\pm j=0} lsm^2 W_{lsmj} z_l \bar{z}_s z_m + i\varepsilon \sum_{l\pm s\pm m\pm j=0} (l^4 - s^4 + m^4 - j^4) T_{lsmj} z_l \bar{z}_s z_m + O(\|\boldsymbol{z}\|_p^5).$$

Using (14), (8) is changed into (12).

Next we prove that the change of variables ψ commutes with the involution G_0, i. e. $\psi\circ G_0 = G_0\circ\psi$.

By the assumption that (8) is reversible with respect to G_0, we obtain that W_{lsmj} is in $\mathbf{R}$. Thus by (14), we obtain $T_{lsmj}\in\mathbf{R}$. Then

$$\psi\circ G_0(\boldsymbol{q},\bar{\boldsymbol{q}}) = \psi(\bar{\boldsymbol{q}},\boldsymbol{q}) = (\bar{\boldsymbol{z}},\boldsymbol{z}) = G_0(\boldsymbol{z},\bar{\boldsymbol{z}}) = G_0\circ\psi(\boldsymbol{q},\bar{\boldsymbol{q}}).$$

This result guarantees that the transformed system (12) is also a reversible system with respect to the involution $G_0:(\boldsymbol{z},\bar{\boldsymbol{z}})\to(\bar{\boldsymbol{z}},\boldsymbol{z})$.

As the coefficients of transformation (13) are real, so the coefficients of high order nonnormal terms for (12) are still imaginary.

2 The proof of main results

In this section, we will present the proof of Theorem 1.

For a complex vector $\boldsymbol{z} = (z_j)_{j\geqslant 1}\in\ell^p$ and a given set $J = \{j_1 < j_2 < \cdots < j_n\} \in \mathbf{N} = \{1,2,\cdots\}$, we write

$$\begin{cases} \tilde{\boldsymbol{z}} = (z_j)_{j\in J}, \\ \hat{\boldsymbol{z}} = (z_j)_{j\in \mathbf{N}_J}, \end{cases} \quad \begin{cases} \boldsymbol{Y} = \left(\dfrac{1}{2}|z_j|^2\right)_{j\in J}^{\mathrm{T}}, \\ \boldsymbol{Z} = \left(\dfrac{1}{2}|z_j|^2\right)_{j\in \mathbf{N}_J}^{\mathrm{T}}, \end{cases}$$

and denote

$$\begin{cases} \boldsymbol{\beta}_1 := (j^4)_{j\in J}, \\ \boldsymbol{\beta}_2 := (j^4)_{j\in\mathbf{N}_J}, \end{cases} \quad \begin{cases} \boldsymbol{A} := (2W_{lj} j^2 l^2)_{j,l\in J}, \\ \boldsymbol{B}_2 := (2W_{lj} j^2 l^2)_{j,l\in\mathbf{N}_J}, \end{cases} \quad \begin{cases} \boldsymbol{B}_1 := (2W_{lj} j^2 l^2)_{j\in\mathbf{N}_J, l\in J}, \\ \boldsymbol{B}_3 := (2W_{lj} j^2 l^2)_{l\in\mathbf{N}_J, j\in J}, \end{cases}$$

So we can rewrite (12) as

$$\begin{cases} \dot{\tilde{\boldsymbol{z}}} = i\boldsymbol{\beta}_1 \tilde{\boldsymbol{z}} - i\varepsilon\,\mathrm{diag}(\boldsymbol{AY} + \boldsymbol{B}_3\boldsymbol{Z})\tilde{\boldsymbol{z}} + i\varepsilon g^1(\tilde{\boldsymbol{z}},\hat{\boldsymbol{z}},\bar{\tilde{\boldsymbol{z}}},\bar{\hat{\boldsymbol{z}}}), \\ \dot{\hat{\boldsymbol{z}}} = i\boldsymbol{\beta}_2 \hat{\boldsymbol{z}} - i\varepsilon\,\mathrm{diag}(\boldsymbol{B}_1\boldsymbol{Y} + \boldsymbol{B}_2\boldsymbol{Z})\hat{\boldsymbol{z}} + i\varepsilon g^2(\tilde{\boldsymbol{z}},\hat{\boldsymbol{z}},\bar{\tilde{\boldsymbol{z}}},\bar{\hat{\boldsymbol{z}}}), \end{cases} \tag{16}$$

which is reversible with respect to the involution $G_0:(\tilde{z},\hat{z},\bar{\tilde{z}},\bar{\hat{z}})\rightarrow(\tilde{z},\bar{\hat{z}},\bar{\tilde{z}},\hat{z})$ and

$$g^1(\tilde{z},\hat{z},\bar{\tilde{z}},\bar{\hat{z}})(g_j(\bar{\tilde{z}},\bar{\hat{z}},\tilde{z},\hat{z}))_{j\in J},g^2(\tilde{z},\hat{z},\bar{\tilde{z}},\bar{\hat{z}})=(g_j(\bar{\tilde{z}},\bar{\hat{z}},\tilde{z},\hat{z}))_{j\in \mathbf{N}_J}.$$

For the parameter $\boldsymbol{\xi}=(\xi_1,\xi_2,\cdots,\xi_n)\in\Theta=\{\boldsymbol{\xi}\in\mathbf{R}^n\mid 0<|\xi_j|<1\}$, we introduce a transformation φ:

$$\begin{cases}\tilde{z}_j=\sqrt{2I_j+\xi_j}\,\mathrm{e}^{i\theta_j},j\in J,\\ \hat{z}_j=z_j,j\in\mathbf{N}_J,\end{cases}\tag{17}$$

which transforms (16) into the following form

$$\begin{cases}\dot{\theta}_j=\omega_j(\xi)-\varepsilon\mathrm{diag}(\boldsymbol{A}\boldsymbol{I}+\boldsymbol{B}_3\boldsymbol{Z})_j+\frac{\varepsilon}{2}\left(\frac{g_j^1}{\tilde{z}_j}+\frac{\bar{g}_j^1}{\bar{\tilde{z}}_j}\right),j\in J,\\ \dot{I}_j=\frac{i\varepsilon}{2}(g_j^1\bar{\tilde{z}}_j-\bar{g}_j^1\tilde{z}_j),j\in J,\\ \dot{z}_j=i\lambda_j(\xi)z_j-i\varepsilon\mathrm{diag}(\boldsymbol{B}_1\boldsymbol{I}+\boldsymbol{B}_2\boldsymbol{Z})_jz_j+i\varepsilon g_j^2\circ\varphi(\theta,I,z,\bar{z},\xi),j\in\mathbf{N}_J,\\ \dot{\bar{z}}_j=-i\lambda_j(\xi)\bar{z}_j+i\varepsilon\mathrm{diag}(\boldsymbol{B}_1\boldsymbol{I}+\boldsymbol{B}_2\boldsymbol{Z})_j\bar{z}_j-i\varepsilon\,\overline{g_j^2\circ\varphi(\theta,I,z,\bar{z},\xi)},j\in\mathbf{N}_J,\end{cases}\tag{18}$$

with $\omega_j(\boldsymbol{\xi})(\boldsymbol{\beta}_1-\boldsymbol{\varepsilon A\xi})_j,\lambda_j(\boldsymbol{\xi})(\boldsymbol{\beta}_2-\boldsymbol{\varepsilon B}_1\boldsymbol{\xi})_j$. Then the tangent and normal frequency of system (18) are

$$\omega(\boldsymbol{\xi})\mathrm{diag}(\omega_j(\boldsymbol{\xi}))_{j\in J},\ \Lambda(\boldsymbol{\xi})\mathrm{diag}(\lambda_j(\boldsymbol{\xi}))_{j\in\mathbf{N}_J}.\tag{19}$$

In the following, we will adopt lots of notations and definitions from [18], which including the phase space, weighted norm for the reversible vector field, etc.. More definitions are presented in [18]. To obtain our result, we have to verify (18) satisfies assumptions (A1)-(A4) of Theorem 1.1 in [18].

Firstly, from (19), we have $|\boldsymbol{\omega}|_{\Theta}^{lip}=|\beta_1-\varepsilon\boldsymbol{A}\xi|_{\Theta}^{lip}\leqslant\varepsilon\frac{2nj_n^4}{\pi}:=M$, where $\boldsymbol{A}=(2W_{lj}j^2l^2)_{j,l\in J}$ and W_{lj} is given as follows $W_{lj}=\frac{1}{\pi^2}\int_0^{\pi}\sin^2 jx\cos^2 lx\,\mathrm{d}x=\frac{1+\delta_{jl}}{2\pi},\delta_{jl}=\begin{cases}0,j\neq l,\\1,j=l.\end{cases}$

Because $|\boldsymbol{A}|=\frac{3}{\pi}\cdot 2^{n-1}j_2^4j_2^4\cdots j_n^4\neq0$, $\omega(\boldsymbol{\xi})$ is a homomorphism and Lipschitz continuous in both directions. Clearly, by (19), $\langle\boldsymbol{l},\beta_2\rangle\neq0$ for $1\leqslant|\boldsymbol{l}|\leqslant2$. To verify condition

$$\mathrm{meas}(\langle\boldsymbol{\xi}:\langle\boldsymbol{k},\omega(\boldsymbol{\xi})\rangle+\langle\boldsymbol{l},\Lambda(\xi)\rangle=0\rangle)=0,\tag{20}$$

we have to check that $\langle\boldsymbol{k},\boldsymbol{\beta}_1\rangle+\langle\boldsymbol{l},\boldsymbol{\beta}_2\rangle\neq0$ or $\boldsymbol{kA}+\boldsymbol{lB}_1\neq0$, for all $(\boldsymbol{k},\boldsymbol{l})\in\mathbf{Z}^n\times\mathbf{Z}^\infty$ with $1\leqslant|\boldsymbol{l}|\leqslant2$.

Suppose that

$$\boldsymbol{kA}+\boldsymbol{lB}_1=0\tag{21}$$

and multiply the matric $\boldsymbol{P}=\mathrm{diag}\left(\frac{1}{j_1^2},\frac{1}{j_2^2},\cdots,\frac{1}{j_n^2}\right)$ from the right-hand side of (21) and we can obtain

$$\boldsymbol{k}\tilde{\boldsymbol{A}}+\boldsymbol{l}\tilde{\boldsymbol{B}}=0,$$

where

$$\tilde{\boldsymbol{A}}=\begin{pmatrix}2j_1^2 & j_1^2 & \cdots & j_1^2\\ j_2^2 & 2j_2^2 & \cdots & j_2^2\\ \cdots & & & \cdots\\ j_n^2 & j_n^2 & \cdots & 2j_n^2\end{pmatrix},\tilde{\boldsymbol{B}}=(\tilde{\boldsymbol{B}}_j)_{j\in\mathbf{N}_J},\tilde{\boldsymbol{B}}_j=(j^2,j^2,\cdots,j^2).$$

Using a series of elementary transformation, we get

$$\tilde{\boldsymbol{A}}^{-1}=\frac{1}{n+1}\begin{pmatrix}\frac{n}{j_1^2} & \frac{-1}{j_2^2} & \cdots & \frac{-1}{j_n^2}\\ \frac{-1}{j_1^2} & \frac{n}{j_2^2} & \cdots & \frac{-1}{j_n^2}\\ \cdots & & & \cdots\\ \frac{-1}{j_1^2} & \frac{-1}{j_2^2} & \cdots & \frac{n}{j_n^2}\end{pmatrix}.$$

Thus, the quality (21) holds true for $\boldsymbol{k}\in\mathbf{Z}^n$ and $\boldsymbol{l}\in\mathbf{Z}^\infty$ with $1\leqslant|\boldsymbol{l}|\leqslant2$ except the following four cases

Case 1 $l_j=\begin{cases}\pm1,j=\sqrt{h(n+1)}[j_1,j_2,\cdots,j_n],\\0,\text{otherwise},\end{cases}$

and $\boldsymbol{k}=\mp h[j_1^2,j_2^2,\cdots,j_n^2]\left(\frac{1}{j_1^2},\frac{1}{j_2^2},\cdots,\frac{1}{j_n^2}\right)$ with $h\in\mathbf{N}$;

Case 2 $l_j=\begin{cases}\pm 2, j=\sqrt{\dfrac{h(n+1)}{2}}[j_1,j_2,\cdots,j_n],\\ 0,\text{otherwise},\end{cases}$

and $\boldsymbol{k}=\mp h[j_1^2,j_2^2,\cdots,j_n^2]\left(\dfrac{1}{j_1^2},\dfrac{1}{j_2^2},\cdots,\dfrac{1}{j_n^2}\right)$ with $h\in\mathbf{N}$;

Case 3 $l_j=\begin{cases}\pm 1, j=j',j'',\text{and } j'^2+j''^2=h(n+1)[j_1,j_2,\cdots,j_n],\\ 0,\text{otherwise},\end{cases}$

and $\boldsymbol{k}=\mp h[j_1^2,j_2^2,\cdots,j_n^2]\left(\dfrac{1}{j_1^2},\dfrac{1}{j_2^2},\cdots,\dfrac{1}{j_n^2}\right)$ with $h\in\mathbf{N}$;

Case 4 $l_j=\begin{cases}\pm 1, j=j',\\ \mp 1, j=j''<j',\text{and } j'^2-j''^2=h(n+1)[j_1,j_2,\cdots,j_n],\\ 0,\text{otherwise},\end{cases}$

and $\boldsymbol{k}=\mp h[j_1^2,j_2^2,\cdots,j_n^2]\left(\dfrac{1}{j_1^2},\dfrac{1}{j_2^2},\cdots,\dfrac{1}{j_n^2}\right)$ with $h\in\mathbf{N}$, where $[j_1,j_2,\cdots,j_n]$ is the least common multiple of $j_1,j_2,\cdots,j_n$.

In the following we will check $\langle\boldsymbol{k},\boldsymbol{\beta}_1\rangle+\langle\boldsymbol{l},\boldsymbol{\beta}_2\rangle\neq 0$ holds true for all the above four cases.

If case 1 holds true, we have

$$\begin{aligned}\langle\boldsymbol{k},\boldsymbol{\beta}_1\rangle+\langle\boldsymbol{l},\boldsymbol{\beta}_2\rangle&=\mp h[j_1^2,j_2^2,\cdots,j_n^2](j_1^2+j_2^2+\cdots+j_n^2)\pm h^2(n+1)^2[j_1^2,j_2^2,\cdots,j_n^2]^2\\&=\mp h[j_1^2,j_2^2,\cdots,j_n^2](j_1^2+j_2^2+\cdots+j_n^2-h(n+1)^2[j_1^2,j_2^2,\cdots,j_n^2]).\end{aligned}$$

In view of

$$j_1^2+j_2^2+\cdots+j_n^2<nj_n^2<(n+1)^2[j_1^2,j_2^2,\cdots,j_n^2]<h(n+1)^2[j_1^2,j_2^2,\cdots,j_n^2],$$

then we obtain $\langle\boldsymbol{k},\boldsymbol{\beta}_1\rangle+\langle\boldsymbol{l},\boldsymbol{\beta}_2\rangle\neq 0$.

If case 2 holds true, we can get

$$\begin{aligned}\langle\boldsymbol{k},\boldsymbol{\beta}_1\rangle+\langle\boldsymbol{l},\boldsymbol{\beta}_2\rangle&=\mp h[j_1^2,j_2^2,\cdots,j_n^2](j_1^2+j_2^2+\cdots+j_n^2)\pm\frac{h^2(n+1)^2}{4}[j_1^2,j_2^2,\cdots,j_n^2]^2\\&=\mp h[j_1^2,j_2^2,\cdots,j_n^2]\left(j_1^2+j_2^2+\cdots+j_n^2-\frac{h(n+1)^2}{4}[j_1^2,j_2^2,\cdots,j_n^2]\right).\end{aligned}$$

Notice that

$$j_1^2+j_2^2+\cdots+j_n^2<nj_n^2<\frac{(n+1)^2}{4}[j_1^2,j_2^2,\cdots,j_n^2]<\frac{h(n+1)^2}{4}[j_1^2,j_2^2,\cdots,j_n^2],$$

then $\langle\boldsymbol{k},\boldsymbol{\beta}_1\rangle+\langle\boldsymbol{l},\boldsymbol{\beta}_2\rangle\neq 0$.

If case 3 holds true, it is easy to see

$$\langle\boldsymbol{k},\boldsymbol{\beta}_1\rangle+\langle\boldsymbol{l},\boldsymbol{\beta}_2\rangle=\mp h[j_1^2,j_2^2,\cdots,j_n^2](j_1^2+j_2^2+\cdots+j_n^2)\pm(j'^4+j''^4).$$

From $(j'^4+j''^4)>\dfrac{1}{2}(j'^2+j''^2)^2$, we have

$$(j'^4+j''^4)>\frac{1}{2}h^2(n+1)^2[j_1^2,j_2^2,\cdots,j_n^2]^2>\frac{1}{2}h(n+1)^2[j_1^2,j_2^2,\cdots,j_n^2]^2.$$

Then

$$\langle\boldsymbol{k},\boldsymbol{\beta}_1\rangle+\langle\boldsymbol{l},\boldsymbol{\beta}_2\rangle=\mp h[j_1^2,j_2^2,\cdots,j_n^2]\left(j_1^2+j_2^2+\cdots+j_n^2-\frac{j'^4+j''^4}{h[j_1^2,j_2^2,\cdots,j_n^2]}\right).$$

Moreover,

$$j_1^2+j_2^2+\cdots+j_n^2<nj_n^2<\frac{(n+1)^2}{2}[j_1^2,j_2^2,\cdots,j_n^2]<\frac{j'^4+j''^4}{h[j_1^2,j_2^2,\cdots,j_n^2]},$$

therefore we can obtain $\langle\boldsymbol{k},\boldsymbol{\beta}_1\rangle+\langle\boldsymbol{l},\boldsymbol{\beta}_2\rangle\neq 0$.

If case 4 holds true, it is easy to see

$$\langle\boldsymbol{k},\boldsymbol{\beta}_1\rangle+\langle\boldsymbol{l},\boldsymbol{\beta}_2\rangle=\mp h[j_1^2,j_2^2,\cdots,j_n^2](j_1^2+j_2^2+\cdots+j_n^2)\pm(j'^4-j''^4).$$

Notice $(j'^4-j''^4)=(j'^2+j''^2)(j'^2-j''^2)>(j'^2-j''^2)^2=h(n+1)^2[j_1^2,j_2^2,\cdots,j_n^2]^2$, we have

$$\langle\boldsymbol{k},\boldsymbol{\beta}_1\rangle+\langle\boldsymbol{l},\boldsymbol{\beta}_2\rangle=\mp h[j_1^2,j_2^2,\cdots,j_n^2]\left(j_1^2+j_2^2+\cdots+j_n^2-\frac{j'^4-j''^4}{h[j_1^2,j_2^2,\cdots,j_n^2]}\right).$$

Moreover,

$$j_1^2+j_2^2+\cdots+j_n^2<nj_n^2<n[j_1^2,j_2^2,\cdots,j_n^2]<h(n+1)^2[j_1^2,j_2^2,\cdots,j_n^2]<\frac{j'^4-j''^4}{h[j_1^2,j_2^2,\cdots,j_n^2]},$$

so we can get $\langle \boldsymbol{k},\boldsymbol{\beta}_1\rangle+\langle \boldsymbol{l},\boldsymbol{\beta}_2\rangle\neq 0$.

Therefore, assumption (A1) is satisfied.

Secondly, as

$$\lambda_j(\xi)j^4-\varepsilon(B_1\xi)_j=j^4-\frac{\varepsilon}{\pi}j^2(j_1^2,j_2^2,\cdots,j_n^2)(\xi_1,\xi_2,\cdots,\xi_n)^{\mathrm{T}},$$

thus

$$\begin{aligned}|\lambda_j(\xi)\lambda_l(\xi)|&=\left|j^4-l^4-\frac{\varepsilon}{\pi}(\sum_{s=1}^{n}j_s^2\xi_s)(j^2-l^2)\right|\\&=|j-l|\left|(j+l)(j^2+l^2)-(j+l)\frac{\varepsilon}{\pi}(\sum_{s=1}^{n}j_s^2\xi_s)\right|\\&=|j-l|\left|(j^3+l^3)+(j+l)\left(jl-\frac{\varepsilon}{\pi}\sum_{s=1}^{n}j_s^2\xi_s\right)\right|\\&\geqslant|j-l|\,|j^3+l^3|,\end{aligned}$$

for $j\neq l,j,l>0$ and ε is small enough. Moreover,

$$|\boldsymbol{\Lambda}(\xi)|_{-2,\Theta}^{lip}=\sup j^{-2}|\varepsilon(B_1\xi)_j|_{\Theta}^{lip}\leqslant\frac{\varepsilon\cdot 2nj_n^2}{2\pi}=C_\lambda^{lip}.$$

Therefore, assumption (A2) is satisfied.

It is sufficient to verify the regularity of $f=(f_1,f_2,if_3,-i\bar{f}_3)$ in (18), where

$$\begin{aligned}f_1&=-\varepsilon(\boldsymbol{A}\boldsymbol{I}+\boldsymbol{B}_3|\hat{z}|^2)+\varepsilon\frac{g^1\bar{\tilde{z}}+\bar{g}^1\tilde{z}}{4(\boldsymbol{I}+\boldsymbol{\xi})},\\f_2&=\frac{i\varepsilon}{2}(g^1\bar{\tilde{z}}-\bar{g}^1\tilde{z}),\\f_3&=-\varepsilon(\boldsymbol{B}_1\boldsymbol{I}+\boldsymbol{B}_2\boldsymbol{Z})\hat{z}+\varepsilon g^2\circ\varphi(\boldsymbol{\theta},\boldsymbol{I},z,\bar{z},\boldsymbol{\xi}),\\\bar{f}_3&=-\varepsilon(\boldsymbol{B}_1\boldsymbol{I}+\boldsymbol{B}_2\boldsymbol{Z})\bar{\hat{z}}+\varepsilon\overline{g^2\circ\varphi(\boldsymbol{\theta},\boldsymbol{I},z,\bar{z},\boldsymbol{\xi})}.\end{aligned}$$

From Lemma 3, we have

$$|g^1(\tilde{z},\hat{z},\bar{\tilde{z}},\bar{\hat{z}})|=O(\|z\|_p^5),\ \|g^2(\tilde{z},\hat{z},\bar{\tilde{z}},\bar{\hat{z}})\|_{p-2}=O(\|z\|_p^5).$$

Now let $0<r<\frac{1}{2}$, and consider the phase space domain

$$D(s,r)=\{|Im\theta|<s,|I|<r^2,\|q\|_p<r,\|\bar{q}\|_p<r\}.$$

Set $\tilde{\Theta}:=\{\boldsymbol{\xi}\in\mathbf{R}^n:\frac{r}{2}<|\xi|<r\}$, we obtain the perturbationfis analytic on $D(s,r)\times\tilde{\Theta}$. So the assumption (A3) is satisfied.

Lastly, It is easy to see that φ satisfies

$$G_0\circ\varphi=\varphi\circ G_1, \tag{22}$$

where $G_0:(\tilde{z},\hat{z},\bar{\tilde{z}},\bar{\hat{z}})\to(\bar{\tilde{z}},\bar{\hat{z}},\tilde{z},\hat{z})$ and $G_1:(\boldsymbol{\theta},\boldsymbol{I},z,\bar{z})\to(-\boldsymbol{\theta},\boldsymbol{I},\bar{z},z)$.

Denote X_{old} and X_{new} as the vector fields of (16) and (18), respectively, as (16) is reversible with respect to G_0, thus we have $X_{\text{old}}\circ G_0=-DG_0\cdot X_{\text{old}}$. By the transformation φ in (17), we have $X_{\text{old}}(\varphi)=-D\varphi\cdot X_{\text{new}}$.

Moreover, from (22), we get $D\varphi(G_1)\cdot DG_1=DG_0(\varphi)\cdot D\varphi$. Therefore, we obtain

$$\begin{aligned}X_{\text{new}}\circ G_1&=(D\varphi)^{-1}\cdot X_{\text{old}}(\varphi\circ G_1)=(D\varphi)^{-1}\cdot X_{\text{old}}(G_0\circ\varphi)\\&=-(D\varphi)^{-1}(DG_0)\cdot X_{\text{old}}(\varphi)=-DG_1\cdot(D\varphi)^{-1}\cdot X_{\text{old}}(\varphi)\\&=-DG_1\cdot X_{\text{new}}.\end{aligned}$$

This is means that (18) is reversible with respect to G. Namely, the assumption (A4) is satisfied.

If we let $r=\varepsilon^{\frac{1}{5}}$, then

$$\|f\|_{r,p-2,D(s,r)\times\tilde{\Theta}}=O(\varepsilon r^2)=O(r^7), \tag{23}$$

$$\|f\|_{r,p-2,D(s,r)\times\tilde{\Theta}}^{lip}=O(r^7\cdot r^{-1})=O(r^6). \tag{24}$$

Choosing $\alpha=r^{\frac{6}{5}}\tilde{\gamma}^{-\frac{1}{5}}$, we know thatfsatisfies the smallness condition

$$\|f\|_{r,p-2,D(s,r)\times\tilde{\Theta}}+\frac{\alpha}{M}\|f\|_{r,p-2,D(s,r)\times\tilde{\Theta}}^{lip}=O(r^7)\leqslant\tilde{\gamma}\alpha^5. \tag{25}$$

It is easy to verify the other conditions and assumptions of Theorem 1.1 in [18] also hold true. The proof of Theorem 1.1 is completed.

3 Appendix: NLS (4) is not Hamiltonian

In this subsection, we will prove that the NLS equation (4) is not an infinite dimensional Hamiltonian system with the usual symplectic structure. We discuss an abstract evolution equation

$$\bar{u}=f(u),u\in O\subset X, \tag{26}$$

where X is a Hilbert space with the standard inner production

$$\langle u,v\rangle=Re\int_0^\pi u\bar{v}\mathrm{d}x$$

and O is an open domain. Suppose that (26) is a Hamiltonian with respect to symplectic form ω with

$$\omega\langle h_1,h_2\rangle=\langle Jh_1,h_2\rangle,h_1,h_2\in X,$$

where J is an anti-symmetric operator. Then we have

$$\langle J^{-1}f(u)*h_1,h_2\rangle=\langle h_1,J^{-1}f(u)*h_2\rangle,h_1,h_2\in X,u\in O, \tag{27}$$

where $f(u)*$ is a derivative of $f(u)$. More details can see [11].

Next we will prove that (4) is not Hamiltonian with symplectic structure $J=i$. To this end, it is enough to prove that $u_t=if(\boldsymbol{u})$ is not Hamiltonian where $u(0)=u(\pi)=0$ and $f(\boldsymbol{u})=|u_x|^2u_{xx}$. Let X be Sobolev space H^p and $O=\{\boldsymbol{u}\in X:\|\boldsymbol{u}\|_p<2\}$. Suppose $u_t=if(\boldsymbol{u})$ is a Hamiltonian system with the usual symplectic form $\omega[h_1,h_2]=[ih_1,h_2]$, noting that

$$(|u_x|^2u_{xx})*h=\lim_{t\to 0}\frac{(u_x+th_x)(\bar{u}_x+t\bar{h}_x)(u_{xx}+th_{xx})-u_x\bar{u}_xu_{xx}}{t}=u_xu_{xx}\bar{h}_x+\bar{u}_xu_{xx}h_x+u_x\bar{u}_xh_{xx},$$

then the left-side of (27) equals

$$\langle -ii(|u_x|^2u_{xx})*h_1,h_2\rangle=\langle u_xu_{xx}\bar{h}_{1x},h_2\rangle+\langle \bar{u}_xu_{xx}h_{1x},h_2\rangle+\langle u_x\bar{u}_xh_{1xx},h_2\rangle, \tag{28}$$

and the right-side of (27) equals

$$\langle h_1,-ii(|u_x|^2u_{xx})*h_2\rangle=\langle h_1,u_xu_{xx}\bar{h}_{2x2}\rangle+\langle h_1,\bar{u}_xu_{xx}h_{2x}\rangle+\langle h_1,u_x\bar{u}_xh_{2xx}\rangle, \tag{29}$$

for any $h_1,h_2\in X$ and $u\in O$.

Choose $0<a\leqslant 1$, and smooth function $u\in O$ and real smooth function $h_2^*(x)\in X$ on $[0,\pi]$ such that

$$Reu_x(u_{xx}+\bar{u}_{xx})(x)=Reu_x(u_{xx}+\bar{u}_{xx})(\pi-x),x\in[0,a],$$

$$Reu_x(u_{xx}+\bar{u}_{xx})(0)=Reu_x(u_{xx}+\bar{u}_{xx})(\pi)=0,$$

$$h_2(x)=ih_2^*(x),h_2^*(x)=h_2^*(\pi-x),x\in[0,a],h_2^*(0)=h_2^*(\pi)=0,$$

and

$$Re\int_a^{\pi-a}u_x(u_{xx}+\bar{u}_{xx})h_{2x}^*\mathrm{d}x\neq 0.$$

Remark 1 There exists function $u(x)$ such that the above condition is satisfied. For example, $u(x)=\sin x+i\sin 2x$. Choose a real smooth function $h_1^*(x)\in X$ such that

$$h_1(x)=ih_1^*(x),h_1^*(x)=h_1^*(\pi-x),x\in[0,a]\cup[\pi-a,\pi],h_1^*(x)=1,x\in(a,\pi-a).$$

Therefore,

$$\begin{aligned}(28)&=Re\int_0^\pi u_xu_{xx}\bar{h}_{1x}\bar{h}_2\mathrm{d}x+Re\int_0^\pi \bar{u}_xu_{xx}h_{1x}\bar{h}_2\mathrm{d}x+Re\int_0^\pi u_x\bar{u}_xh_{1xx}\bar{h}_2\mathrm{d}x\\&=Re\int_0^\pi u_x(u_{xx}+\bar{u}_{xx})h_{1x}^*h_2^*\,\mathrm{d}x-Re\int_0^\pi|u_x|^2h_{1x}^*h_{2x}^*\mathrm{d}x,\\(29)&=Re\int_0^\pi u_xu_{xx}h_{2x}\bar{h}_1\mathrm{d}x+Re\int_0^\pi \bar{u}_xu_{xx}h_{2x}\bar{h}_1\mathrm{d}x+Re\int_0^\pi u_x\bar{u}_xh_{2xx}\bar{h}_1\mathrm{d}x\\&=Re\int_0^\pi u_x(u_{xx}+\bar{u}_{xx})h_1^*\,h_{2x}^*\mathrm{d}x-Re\int_0^\pi|u_x|^2h_{1x}^*h_{2x}^*\mathrm{d}x.\end{aligned}$$

Notice that the second part of (28) and (29) is equal, so we consider the first part of them. Obviously,

$$Re\int_{\pi-a}^\pi u_x(u_{xx}+\bar{u}_{xx})h_{1x}^*h_2^*\,\mathrm{d}x=-Re\int_0^a u_x(u_{xx}+\bar{u}_{xx})h_{1x}^*h_2^*\,\mathrm{d}x,$$

then we know $Re\int_0^\pi u_x(u_{xx}+\bar{u}_{xx})h_{1x}^*h_2^*\,\mathrm{d}x=0$.

Moreover,

$$Re\int_{\pi-a}^\pi u_x(u_{xx}+\bar{u}_{xx})h_1^*\,h_{2x}^*\mathrm{d}x=-Re\int_0^a u_x(u_{xx}+\bar{u}_{xx})h_1^*\,h_{2x}^*\mathrm{d}x,$$

then

$$Re\int_0^\pi u_x(u_{xx}+\bar{u}_{xx})h_1^*\,h_{2x}^*\mathrm{d}x=Re\int_a^{\pi-a}u_x(u_{xx}+\bar{u}_{xx})h_{2x}^*\mathrm{d}x\neq 0,$$

That is, (28) is not equal to (29). Consequently, the system (4) is not Hamiltonian.

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具无界扰动的4阶非线性薛定谔方程的不变环面

崔文艳　弭鲁芳　由红连

(滨州学院 理学院,山东 滨州 256600)

摘　要　考虑了狄利克雷边条件下的四阶非线性薛定谔方程 $iu_t+u_{xxxx}+|u_x|^2u_{xx}=0$. 利用一个无穷维KAM定理,证明上述方程存在大量的 n-不变环面,从而得到方程存在大量的时间拟周期解.

关键词　薛定谔方程;KAM定理;拟周期解;标准型