第 31 卷　第 1 期
2018 年 3 月

聊城大学学报(自然科学版)
Journal of Liaocheng University(Nat. Sci.)

Vol. 31 No. 1
Mar. 2018

Article ID　1672-6634(2018)01-0006-09　　**DOI**　10.19728/j.issn1672-6634.2018.01.002

Positive Solutions for Eigenvalue Problems on a Measure Chain[①]

WANG Zeng-gui

(School of Mathematical Sciences, Liaocheng University, Liaocheng 252059, China)

Abstract　In this paper, we investigate the existence of positive solutions for eigenvalue problems by the fixed point index theory. Our result improves and extends many recent results.

Key words　positive solutions; eigenvalue problems; fixed point index; measure chain

CLC number　O175.8　　**Document code**　A

1　Introduction

In this paper, we consider the eigenvalue problem to the following second order dynamic equation:

$$(p(t)u^{\Delta}(t))^{\Delta}+\lambda g(t)f(t,u(\sigma(t)))=0,\ t\in J, \tag{1}$$

with inhomogeneous boundary conditions

$$\begin{cases}\alpha u(0)-\beta\lim\limits_{t\to 0^+}p(t)u^{\Delta}(\sigma(0))=0,\\ \gamma u(1)-\delta\lim\limits_{t\to 1^-}p(t)u^{\Delta}(\sigma(1))=0.\end{cases} \tag{2}$$

where $J:=[0,1]\cap T$ and $T\subset \mathbf{R}$ is an arbitrary bounded time scale such that $\min T=0$, $\max T=\sigma(1)$, and $\alpha,\beta,\gamma,\delta\geqslant 0$, and it satisfies the following property:

$$\rho=\beta\gamma+\alpha\gamma\int_0^{\sigma(1)}\frac{\Delta r}{p(r)}+\alpha\delta>0.$$

We always assume that the following hypothesis holds

(A_1) $p:[0,\sigma(1)]\to(0,+\infty)$ is rd-continuous function on T, and $\int_0^{\sigma(1)}\frac{\Delta t}{p(t)}<\infty$,

(A_2) $g:[0,\sigma(1)]\to[0,+\infty)$, $g\in L^1_{\Delta}[0,\sigma(1)]$, and $0<\int_0^{\sigma(1)}G(t,s)g(s)\Delta s<+\infty$,

furthermore, there exists $c,d\in[0,\sigma(1)]$ such that

$$\int_c^d g(s)\Delta s>0,$$

(A_3) $f(t,u):[0,\sigma(1)]\times(0,+\infty)\to[0,+\infty)$ is continuous.

For every $t_1,t_2\in T$ such that $t_1<t_2$ and for every $x\in L^1_{\Delta}([t_1,t_2)\cap T)$ we denote

$$\int_{t_1}^{t_2}x(s)\Delta s=\int_{[t_1,t_2)\cap T}x(s)\Delta s,$$

---

① 收稿日期:2017-10-10

基金项目:国家自然科学基金项目(1100115,11201473); 山东省自然科学基金项目(ZR2015AL008)资助

通讯作者:王增桂,女,汉族,博士,副教授,研究方向:偏微分方程、几何分析等,E-mail:wangzenggui@lcu.edu.cn.

as $AC([t_1, t_2) \cap T)$ the class of absolutely continuous on $[t_1, t_2) \cap T$, which are precisely those for which the Fundamental Theorem of Calculus for Lebesgue Δ- integrable functions holds[2, Theorem 4.1], and we remark that they satisfy the integration by parts formula [3, Theorem 2.3].

Equation (1) and (2) describe many phenomena in the applied mathematical sciences, which can be found in the theory of nonlinear diffusion generated by nonlinear sources, in thermal ignition of gases and in concentration in chemical or biological problems where only positive solutions are meaningful, see [2,8,10,13]. In the continuous case equation (1) and (2) (here $\beta=\delta=0$) is investigated by the authors[9,14].

For the special case as follows with $p(t)=1$ and $f(t,u)=f(u)$,

$$u^{\Delta\Delta}(t)+\lambda g(t) f(u(\sigma(t)))=0,\ t \in (0,1), \tag{3}$$

Chyan and Henderson[5] consider the existence of positive solutions of Eq. (3) with the conjugate boundary value condition $u(0)=u(\sigma(1))=0$, or the right focal boundary value condition $u(0)=u^{\Delta}(\sigma(1))=0$.

They obtained excellent results.

When $p(t)\equiv 1, g(t)\equiv 1$, the existence of positive solutions of Eq. (1) and (2) has been studied in [4] by using a fixed point theorem in a cone due to Krasnoselskii[11], they proved that Eq. (1) and (2) has a positive solution where λ belongs to some open interval. The results in [4] generalize the main results in [5].

We remark that by a solution u of (1) and (2), we mean that $u:[0,\sigma^2(1)]\to(-\infty,+\infty)$, u satisfies (1) on $[0,1]\cap T$ and u satisfies (2). We further remark that if u is a nonnegative solution of Eq. (1)—(2), $p(t)u^{\Delta}(t)\leqslant 0$, then we will say u is concave on $[0,\sigma^2(1)]$.

In this paper, we improve on previous results in several ways. We consider Eq. (1) and (2) when $g(t)$ has singularities at $t=0$ and $t=\sigma(1)$. The existence of positive solutions for Eq. (1) and (2) is obtained by using a fixed point index theory under weaker conditions than those in [4,5]. Our results are often new even when $g(t)$ is continuous.

This paper is organized as follows. we first introduce some preliminaries in section 2, then we state our main results in section 3.

2 Preliminaries

In order to discuss Eq. (1) and (2), we first give some definitions of a measure chain, see[1,3,12].

Definition 1 A measure chain T is a closed subset of the set $\mathbf{R}$ of all real numbers. We assume throughout this paper that T has the topology that it inherits from standard topology on $\mathbf{R}$. The mapping $\sigma,\rho: T\to T$ defined as $\sigma(t)=\inf\{s\in T: s>t\}$ and $\rho(t)=\sup\{s\in T: s>t\}$ are called jump operators.

If $\sigma(t)>t, t\in T$, we say t is right-scattered. If $\rho(t)<t, t\in T$, we say t is left-sattered. If $\sigma(t)=t, t\in T$, we say t is right-dense. If $\rho(t)=t, t\in T$, we say t is left-dense.

Definition 2 If $r,s\in T\cup\{+\infty,-\infty\}, r<s$, then an open interval (r, s) in T is defined by

$$(r,s):=\{t\in T: r<t<s\}.$$

Throughout this paper we make the assumption that $[a,b]$ as $[a,b]\cap T$ if $a,b\in\mathbf{R}, a\leqslant b$.

Definition 3 Assume that $f: T\to\mathbf{R}$ and fix $t\in T$ (if $t=\sup T$, we assume t is not left-scattered). Then f is said to be differentiable at $t\in T$, if there exists a $\theta\in\mathbf{R}$, such that for any $\varepsilon>0$ there exists a neighborhood N of t satisfying $|f(\sigma(t))-f(s)-\theta(\sigma(t)-s)|\leqslant\varepsilon|\sigma(t)-s|, \forall s\in N$.

In this case, θ is called the Δ- derivative of f at $t\in T$, and denote $f^{\Delta}(t)=\theta$.

It can be shown that if $f: T\to\mathbf{R}$ is continuous at $t\in T$, then

(i) it is continuous at each right-dense $t \in T$,

(ii) it is continuous at each left-dense $t \in T$.

Definition 4 If $F^{\Delta}(t) = f(t)$, then we define $\int_a^t f(\tau)\Delta\tau = F(t) - F(a)$, it can be shown that if $f: T \to \mathbf{R}$ is continuous at $t \in T$ and t is right-scattered, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

Note that if $T = Z$ then $f^{\Delta}(t) = \Delta f(t) := f(t+1) - f(t)$,

if t is right-dense, then

$$f^{\Delta}(t) = \lim_{t \to s} \frac{f(s) - f(t)}{s - t}.$$

also an integral $\int_a^b h(t)\Delta t$ can be defined. In particular, if $T = \mathbf{R}$ then $f^{\Delta}(t) = f'(t)$, and

$$\int_a^b h(t)\Delta t = \int_a^b h(t)\mathrm{d}t$$

is the Riemann integral. If $T = Z$ and $a < b$ are integers, then

$$\int_a^b h(t)\Delta t = \sum_{t=a}^{b-1} h(t).$$

Next we gather together some results from the literature that will be needed in Section 3. similar to the proof of Erbe and Peterson[6,7], we show that the Green's function for the boundary value problem

$$\begin{cases} (p(t)u^{\Delta}(t)) = 0, \\ \alpha u(0) - \beta \lim\limits_{t \to 0^+} p(t)u^{\Delta}(\sigma(0)) = 0, \\ \gamma u(1) - \delta \lim\limits_{t \to 1^-} p(t)u^{\Delta}(\sigma(1)) = 0, \end{cases} \tag{4}$$

is the following function

$$G(t,s) = \begin{cases} \frac{1}{\rho}(\beta + \alpha \int_0^s \frac{\Delta r}{p(r)})(\delta + \gamma \int_t^{\sigma(1)} \frac{\Delta r}{p(r)}), 0 \leqslant s \leqslant t \leqslant \sigma(1), \\ \frac{1}{\rho}(\beta + \alpha \int_0^t \frac{\Delta r}{p(r)})(\delta + \gamma \int_s^{\sigma(1)} \frac{\Delta r}{p(r)}), 0 \leqslant t \leqslant s \leqslant \sigma(1). \end{cases} \tag{5}$$

In order to abbreviate our discussion, we list some assumptions on $m, G(t,s), f$ and so on as follows:

(C_1) $\xi := \min\{t \in T : t \geqslant c\}$ and $\omega := \min\{t \in T : t \leqslant d\}$ both exist and satisfy $c \leqslant \xi \leqslant \omega \leqslant d$, where c, d are defined in (A_2).

(C_2) $m = \min\{d_0, l\}$, where

$$d_0 = \min\left\{\frac{\delta + \gamma \int_d^{\sigma(1)} \frac{\Delta r}{p(r)}}{\delta + \gamma \int_0^{\sigma(1)} \frac{\Delta r}{p(r)}}, \frac{\beta + \alpha \int_0^c \frac{\Delta r}{p(r)}}{\beta + \alpha \int_0^{\sigma(1)} \frac{\Delta r}{p(r)}}\right\}, 0 < d_0 < 1,$$

and

$$l = \min_{s \in [0, \sigma(1)]} \frac{G(\sigma(\omega), s)}{G(\sigma(s), s)}.$$

(C_3) Let $m_1 = (\min\limits_{\xi \leqslant t \leqslant \omega} \int_{\xi}^{\omega} G(t,s)g(s)\Delta s)^{-1}, M_1 = (\min\limits_{0 \leqslant t \leqslant \sigma(1)} \int_0^{\sigma(1)} G(t,s)g(s)\Delta s)^{-1}$.

(C_4) There exist nonnegative constants $\max f_0, \min f_0, \max f_{\infty}, \min f_{\infty}$ such that

$$\max f_0 := \limsup_{u \to 0^+} \max_{t \in [0, \sigma(1)]} \frac{f(t,u)}{u}, \min f_0 := \liminf_{u \to 0^+} \min_{t \in [0, \sigma(1)]} \frac{f(t,u)}{u},$$

$$\max f_{\infty} := \limsup_{u \to \infty} \max_{t \in [0, \sigma(1)]} \frac{f(t,u)}{u}, \min f_{\infty} := \liminf_{u \to \infty} \min_{t \in [0, \sigma(1)]} \frac{f(t,u)}{u}.$$

Remark 1 For condition (C_4), if $\max f_\infty = \min f_0 = \infty, \max f_\infty = \min f_\infty = 0$, then f is said to be sublinear.

Lemma 1 Suppose that $G(t,s)$ is defined as in (5), then the following results hold

(i) $G(t,s) \leqslant G(\sigma^2(s),s)$ for $t \in [0,\sigma^2(1)]$ and $s \in [0,\sigma(1)]$,

(ii) $G(t,s) \geqslant mG(\sigma(s),s)$ for $t \in [\xi,\omega]$ and $s \in [0,\sigma(1)]$,

where m is defined as in (C_2).

Remark 2 By Lemma 1, it is easy to see

$$0 < \min_{\xi \leqslant t \leqslant \omega} \int_\xi^\omega G(t,s)g(s)\Delta s < \infty, 0 < \min_{0 \leqslant t \leqslant \sigma(1)} \int_0^{\sigma(1)} G(t,s)g(s)\Delta s < \infty.$$

Hence m_1, M_1 which are defined as in (C_3) exist.

Let $X := \{u: [0,\sigma^2(1)] \to \mathbf{R}\}$, for all $u \in X$, u is continuous, then X is a Banach space with the norm $\| u \| = \sup_{t \in [0,\sigma^2(1)]} | u(t) |$. Defined a subset of X, $K = \{u \in X: u(t) \geqslant 0, t \in [0,\sigma^2(1)], \min_{\xi \leqslant t \leqslant \omega} | u(t) | \geqslant m \| u \| \}$.

Then K be a cone in a Banach space X and let $K_r = \{x \in K: \| x \| < r\}$, $\partial K_r = \{x \in K: \| x \| = r\}$ and $\overline{K_{r,R}} = \{x \in K: r \leqslant \| x \| \leqslant R\}$, where $0 < r < R < +\infty$.

Define an operator A by

$$(Au)(t) = \lambda \int_0^{\sigma(1)} G(t,s)g(s)f(s,u(\sigma(s)))\Delta s. \tag{6}$$

It is well known that $u \in X$ is a positive solution of Eq. (1) and (2) if and only if u is a fixed point of the operator A in K.

Lemma 2 Under the hypotheses (A_1)-(A_3), the map A defined in (5) is a compact map, and $AK \subset K$.

Proof Let $D \subset K$ be bounded, i. e., $\| u \| \leqslant L$ for all $u \in D$ and some $L > 0$, Let $M = \max\{f(s,u): 0 \leqslant u \leqslant L\}$. It is clear that if $u \in D$, we have

$$\begin{aligned} \| Au \| &= \lambda \sup_{0 \leqslant t \leqslant \sigma^2(1)} \int_0^{\sigma(1)} G(t,s)g(s)f(s,u(\sigma(s)))\Delta s \\ &\leqslant \lambda M \sup_{0 \leqslant t \leqslant \sigma^2(1)} \int_0^{\sigma(1)} G(t,s)g(s)\Delta s < +\infty, \end{aligned} \tag{7}$$

So $A(D)$ is uniformly bounded.

Next we prove that $| (Au)^\Delta(t) | \in L^1_\Delta(0,\sigma(1))$ for $u \in D$. In fact, for $u \in D$.

$$\begin{aligned} | (Au)^\Delta(t) | &= \lambda \Big| -\frac{\gamma}{\rho} \int_0^t (\beta + \alpha \int_0^s \frac{\Delta r}{p(r)}) \frac{1}{p(t)} g(s)f(s,u(\sigma(s)))\Delta s \\ &\quad + \frac{\alpha}{\rho} \int_t^{\sigma(1)} (\delta + \gamma \int_s^{\sigma(1)} \frac{\Delta r}{p(r)}) \frac{1}{p(t)} g(s)f(s,u(\sigma(s)))\Delta s \Big| \\ &\leqslant \frac{\lambda\gamma}{\rho} \int_0^t (\beta + \alpha \int_0^s \frac{\Delta r}{p(r)}) \frac{1}{p(t)} g(s)f(s,u(\sigma(s)))\Delta s \\ &\quad + \frac{\lambda\alpha}{\rho} \int_t^{\sigma(1)} (\delta + \gamma \int_s^{\sigma(1)} \frac{\Delta r}{p(r)}) \frac{1}{p(t)} g(s)f(s,u(\sigma(s)))\Delta s \\ &\leqslant \lambda M q(t), \end{aligned} \tag{8}$$

in which

$$q(t) = \frac{\alpha}{\rho} \int_t^{\sigma(1)} (\delta + \gamma \int_s^{\sigma(1)} \frac{\Delta r}{p(r)}) \frac{1}{p(t)} g(s)\Delta s + \frac{\gamma}{\rho} \int_0^t (\beta + \alpha \int_0^s \frac{\Delta r}{p(r)}) \frac{1}{p(t)} g(s)\Delta s, \tag{9}$$

and

$$
\begin{aligned}
\int_0^{\sigma(1)} |q(t)| \Delta t &= \int_0^{\sigma(1)} \frac{\gamma}{\rho p(t)} \Delta t \int_0^t (\beta + \alpha \int_0^s \frac{\Delta r}{p(r)}) g(s) \Delta s \\
&\quad + \int_0^{\sigma(1)} \frac{\alpha}{\rho p(t)} \Delta t \int_t^{\sigma(1)} (\delta + \gamma \int_s^{\sigma(1)} \frac{\Delta r}{p(r)}) g(s) \Delta s \\
&= \int_0^{\sigma(1)} \frac{\gamma}{\rho} (\beta + \alpha \int_0^s \frac{\Delta r}{p(r)}) g(s) \Delta s \int_s^{\sigma(1)} \frac{1}{p(t)} \Delta t \\
&\quad + \int_0^{\sigma(1)} \frac{\alpha}{\rho} (\delta + \gamma \int_s^{\sigma(1)} \frac{\Delta r}{p(r)}) g(s) \Delta s \int_0^s \frac{1}{p(t)} \Delta t \\
&= \int_0^{\sigma(1)} \frac{\gamma}{\rho} \int_s^{\sigma(1)} \frac{1}{p(r)} \Delta r (\beta + \alpha \int_0^s \frac{\Delta r}{p(r)}) g(s) \Delta s \\
&\quad + \int_0^{\sigma(1)} \frac{\alpha}{\rho} \int_0^s \frac{1}{p(r)} \Delta r (\delta + \gamma \int_s^{\sigma(1)} \frac{\Delta r}{p(r)}) g(s) \Delta s \\
&\leqslant 2 \int_0^{\sigma(1)} \frac{1}{\rho} (\beta + \alpha \int_0^s \frac{\Delta r}{p(r)}) (\delta + \gamma \int_s^{\sigma(1)} \frac{\Delta r}{p(r)}) g(s) \Delta s \\
&\leqslant 2 \int_0^{\sigma(1)} \frac{1}{\rho} (\beta + \alpha \int_0^{\sigma(1)} \frac{\Delta r}{p(r)}) (\delta + \gamma \int_0^{\sigma(1)} \frac{\Delta r}{p(r)}) g(s) \Delta s \\
&= 2 \int_0^{\sigma(1)} G(\sigma(s), s) g(s) \Delta s \\
&< \infty,
\end{aligned}
\tag{10}
$$

From (6) and (7), we get

$$
0 \leqslant \int_0^{\sigma^2(1)} |(Au)^{\Delta}(t)| \Delta t < \infty, \tag{11}
$$

Therefore, for any $0 \leqslant t_1 \leqslant t_2 \leqslant \sigma^2(1), u \in D$, we have

$$
|Au(t_1) - Au(t_2)| = \left| \int_{t_1}^{t_2} (Au)^{\Delta}(t) \Delta t \right| < \int_{t_1}^{t_2} |(Au)^{\Delta}(t)| \Delta t. \tag{12}
$$

By the absolute continuity of the integral and (6)and (7), it is easy to find that $A(D)$ is equicontinuous. From the Ascoli-Arzela theorem, $A(D)$ is relatively compact.

Next if $u \in K$, it follows from (i) of Lemma 1 that

$$
\begin{aligned}
Au(t) &= \lambda \int_0^{\sigma(1)} G(t,s) g(s) f(s, u(\sigma(s))) \Delta s \\
&\leqslant \lambda \int_0^{\sigma(1)} G(s,s) g(s) f(s, u(\sigma(s))) \Delta s,
\end{aligned}
\tag{13}
$$

thus

$$
\| Au \| \leqslant \lambda \int_0^{\sigma(1)} G(s,s) g(s) f(s, u(\sigma(s))) \Delta s, \tag{14}
$$

If $u \in K$, then by (4), (11) and (ii) of Lemma 1

$$
\begin{aligned}
\min_{\xi \leqslant t \leqslant \omega} Au(t) &= \lambda \int_0^{\sigma(1)} G(t,s) g(s) f(s, u(\sigma(s))) \Delta s \\
&\geqslant m\lambda \int_0^{\sigma(1)} G(\sigma(s), s) g(s) f(s, u(\sigma(s))) \Delta s \\
&\geqslant m \| Au \|,
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
Au(\sigma(\omega)) &= \lambda \int_0^{\sigma(1)} G(\sigma(\omega), s) g(s) f(s, u(\sigma(s))) \Delta s \\
&\geqslant m\lambda \int_0^{\sigma(1)} G(s,s) g(s) f(s, u(\sigma(s))) \Delta s \\
&\geqslant m \| Au \|,
\end{aligned}
\tag{16}
$$

Thus $Au \in K$, if $u \in K$, and hence, $AK \subset K$.

Finally, we prove that A is continuous. It is easy to see [10] that $\| u_n - u \| \to 0$ implies that

$$\rho_n = \sup_{s\in[0,\sigma(1)]} | f(s,u_n(\sigma(s))) - f(s,u(\sigma(s))) | \to 0, \tag{17}$$

as $n\to\infty$. This together with [10, Theorem1. 4. 3(iii)] gives for $t\in[0,\sigma^2(1)]$,

$$\begin{aligned} | Au_n(t) - Au(t) | &= | \lambda \int_0^{\sigma(1)} G(t,s)g(s)[f(s,u_n(\sigma(s))) - f(s,u(\sigma(s)))\Delta s | \\ &\leqslant \rho_n\lambda \sup_{t\in[0,\sigma^2(1)]} \int_0^{\sigma(1)} G(t,s)g(s)\Delta s, \end{aligned} \tag{18}$$

It follows from that Lebesgue's theorem that

$$\| Au_n - Au \| \to 0, \tag{19}$$

as $n\to\infty$uniformly in $t\in[0,\sigma^2(1)]$, which shows that A is continuous. Hence we can conclude that the operator A is a compact map. This completes the proof.

We shall need the following well—known result (see, for example, Theorem 12. 3 in [12]).

Let $K_r = \{x\in K: \| x \| < r\}, \partial K_r = \{x\in K: \| x \| = r\}$ and $\overline{K_{r,R}} = \{x\in K: r < \| x \| \leqslant R\}$, $0<r<R<\infty$.

Lemma 3　Let K be a cone in a Banach space X and $A:\overline{K_R}\to K$ a compact map. Assume that the following conditions hold

(i)　$\| Ax \| \leqslant \| x \|$ for $x\in\partial\overline{K_R}$,

(ii)　There exists $e\in\partial K_1$ such that $x\neq Ax+\lambda e$ for $x\in\partial K_r$and $\lambda>0$. Then A has a fixed point in $\overline{K_{r,R}}$.

The same conclusion remains valid if (i) holds on ∂K_r and (ii) holds on ∂K_R.

3　Main results

In this section, by using Lemma 3 we will establish the existence of one positive solution of Eq. (1) and (2).

Theorem 1　Assume that (C_1)-(C_4) hold, if $\min f_\infty>0, \max f_0<\infty, \frac{m_1}{\min f_\infty}<\frac{M_1}{\min f_0}$, Then for any $\lambda\in(\frac{m_1}{\min f_\infty},\frac{M_1}{\max f_0})$, Eq. (1) and (2) has at least one positive solution $u^*(t)$ which belongs to K.

Proof　First, we show that

$$\| Au \| < \| u \|,\ u\in\partial K_r. \tag{20}$$

For all $\lambda\in(\frac{m_1}{\min f_\infty},\frac{M_1}{\max f_0})$, then $\max f_0<\frac{M_1}{\lambda}$, there exists $r>0, \varepsilon_0>0, M_1-\varepsilon_0>0$ such that $f(t,u)\leqslant\frac{1}{\lambda}(M_1-\varepsilon_0)u\leqslant\frac{1}{\lambda}(M_1-\varepsilon_0)r$,　$0\leqslant u\leqslant r$.

For every $u\in\partial K_r$, we have

$$\begin{aligned} \| Au \| &= \max_{t\in[0,\sigma(1)]} \lambda\int_0^{\sigma(1)} G(t,s)g(s)f(s,u(\sigma(s)))\Delta s \\ &\leqslant \frac{1}{\lambda}(M_1-\varepsilon_0)r \max_{t\in[0,\sigma(1)]} \lambda\int_0^{\sigma(1)} G(t,s)g(s)\Delta s \\ &\leqslant r-\varepsilon_0 r \max_{t\in[0,\sigma(1)]}\int_0^{\sigma(1)} G(t,s)g(s)\Delta s \\ &< r = \| u \|. \end{aligned}$$

Hence (20) is satisfied, i. e. , the condition (i) of Lemma 2 holds.

Next by $\frac{m_1}{f_\infty}<\lambda$, there exists $\eta>mr, \varepsilon_1>0$ such that $f(t,u)\geqslant\frac{1}{\lambda}(m_1+\varepsilon_1)u$, $u\geqslant\eta$.

Choose $R = \max\{m^{-1}\eta, r+1\}$, then we have $\min\limits_{\xi\leqslant t\leqslant\omega} u(t) \geqslant m\|u\| \geqslant \eta$, *for* $u \in \partial K_R$. Let $e \equiv 1$, then $e \in \partial K_1$, we prove that $u \neq Au + \mu e$, $u \in \partial K_R$ and $\mu > 0$.

In fact, if not, there are $\mu_0 > 0$ and $u_0 \in \partial K_R$ such that $u_0 = Au_0 + \mu_0 e$.

Let $\varnothing = \min\{u(t): \xi \leqslant t \leqslant \eta\}$, we have

$$\begin{aligned} u_0(t) &= Au_0(t) + \mu_0 e \\ &= \lambda \int_0^{\sigma(1)} G(t,s)g(s)f(s,u(\sigma(s)))\Delta s + \mu_0 e \\ &\geqslant (m_1+\varepsilon_1)\varnothing \int_\xi^\omega G(t,s)g(s)\Delta s + \mu_0 \\ &\geqslant (\varnothing + \mu_0) + \varepsilon_1 \varphi \int_\xi^\omega G(t,s)g(s)\Delta s \\ &> \varnothing + \mu_0, \end{aligned}$$

which contradicts the definition of $\varnothing$. It follows from Lemma 3 that A has a fixed point $u^*(t)$ in $\overline{K_{r,R}}$, i.e., $r < \|u^*\| \leqslant R, u^*(t) \geqslant 0$ on $[0,\sigma^2(1)]$.

Next, we prove $u^*(t) > 0, t \in [0,\sigma^2(1)]$. For any $t \in [0,\sigma^2(1)]$,

$$\begin{aligned} u^*(t) &= Au^*(t) \\ &= \lambda \int_0^{\sigma(1)} G(t,s)g(s)f(s,u^*(\sigma(s)))\Delta s \\ &\geqslant m\lambda \int_0^{\sigma(1)} G(\sigma(s),s)g(s)f(s,u^*(\sigma(s)))\Delta s \\ &\geqslant m\|Au^*\| = m\|u^*\| > 0. \end{aligned}$$

This completes the proof.

Theorem 2 Assume that (C_1)-(C_4) hold, if $\min f_0 > 0, \max f_\infty < \infty, \frac{m_1}{\min f_0} < \frac{M_1}{\min f_\infty}$. Then for any $\lambda \in (\frac{m_1}{\min f_0}, \frac{M_1}{\max f_\infty})$, Eq. (1) and (2) has at least one positive solution $u^{**}(t)$ which belongs to K.

Proof Firstly, $\lambda \in (\frac{m_1}{\min f_0}, \frac{M_1}{\max f_\infty})$, then $\max f_\infty < \frac{M_1}{\lambda}$, there exists $R_0 > 0, \varepsilon_2 > 0, M_1 - \varepsilon_2 > 0$. such that

$$f(t,u) \leqslant \frac{1}{\lambda}(M_1 - \varepsilon_2)u, \quad u \geqslant R_0.$$

Choose $\psi = \max\{f(t,u): 0 \leqslant u_0 \leqslant R_0\}$, then

$$f(t,u) \leqslant \psi + \frac{1}{\lambda}(M_1 - \varepsilon_2)u, (t,u) \in [0,\sigma^2(1)] \times [0,R_0].$$

Choose $R_3 > \psi\varepsilon_2^{-1}\frac{M_1}{\max f_\infty}$, then for any $u \in \partial R_3$, we have

$$\begin{aligned} \|Au\| &= \sup_{t\in[0,\sigma^2(1)]} \lambda \int_0^{\sigma(1)} G(t,s)g(s)f(s,u(\sigma(s)))\Delta s \\ &\leqslant \lambda[\psi + \frac{1}{\lambda}(M_1 - \varepsilon_0)\|u\|] \max_{t\in[0,\sigma^2(1)]} \int_0^{\sigma(1)} G(t,s)g(s)\Delta s \\ &= R_3 - (\frac{\varepsilon_2}{M_1}R_3 - \frac{\psi}{\max f_\infty}) \\ &< R_3 = \|u\|. \end{aligned}$$

Hence the condition (i) of Lemma 3 holds.

Secondly, for all $\lambda \in (\frac{m_1}{\min f_0}, \frac{M_1}{\max f_\infty}), \frac{m_1}{\lambda} < \min f_0$. By the definition of $\min f_0$, there exists $\varepsilon_3 > 0$,

$0 < r_3 < R_3$ such that

$$\frac{m_1}{\min f_0} + \varepsilon_3 < f_0.$$

Hence for $0 < u < r_3$, we have $f(t,u) \geqslant (\frac{m_1}{\lambda} + \varepsilon_3)u$. Similar to the proof of Theorem 1, we have

$$u \neq Au + \mu e, \text{for} \quad u \in \partial K_{r_3} \text{ and } \quad \mu > 0,$$

in which e is defined as in Theorem 1. It follows from Lemma 3 that A has a fixed point $u^{**}(t)$ in $\overline{K_{r_3,R_3}}$, i. e., $r_3 < \| u^* \| \leqslant R_3, u^*(t) \geqslant 0$ on $[0,\sigma^2(1)]$. Similar to the proof in Theorem 3, $u^{**}(t)$ is a positive solution of Eq. (1) and (2). This completes the proof.

Remark 3 In the proof of Theorem 1 and 2 one of the key steps is to find the function e. We note that it seems to be difficult to prove the theorem by using norm-type cone expansion and compression theorem in [4].

Remark 4 As f is suplinear or sublinear, for any $0<\lambda<+\infty$, Eq. (1) and (2)has at least one positive solution which belongs to K.

If $p(t) \equiv 1, g(t) \equiv 1, c = \frac{1}{4}\sigma(1), d = \frac{3}{4}\sigma(1)$, similar to the existence results of the previous section, we can get the following two Theorems

Theorem 3 Assume that (C_1)-(C_4) hold, if

$$\min f_\infty > 0, \max f_0 < \infty, \frac{m'_1}{\min f_\infty} < \frac{M'_1}{\min f_0},$$

where $m'_1 = (\min\limits_{\xi \leqslant t \leqslant \omega} \int_\xi^\omega G(t,s)\Delta s)^{-1}, M'_1 = (\min\limits_{0 \leqslant t \leqslant \sigma^2(1)} \int_0^{\sigma(1)} G(t,s)\Delta s)^{-1}$. Then for any $\lambda \in (\frac{m'_1}{\min f_\infty}, \frac{M'_2}{\max f_0})$, Eq. (1) and (2) has at least one positive solution.

Theorem 4 Assume that (C_1)-(C_4) hold, if

$$\min f_0 > 0, \max f_\infty < \infty, \frac{m'_1}{\min f_0} < \frac{M'_2}{\min f_\infty}.$$

Then for any $\lambda \in (\frac{m'_1}{\min f_0}, \frac{M'_2}{\max f_\infty})$, Eq. (1) and (2) has at least one positive solution.

Remark 5 In Theorem 3.1 of [4],the interval

$$\left(\frac{1}{M(\min f_\infty) \max\limits_{\xi \leqslant t \leqslant \omega} \int_\xi^\omega G(t,s)\Delta s}, \frac{1}{\max f_0 \int_0^1 G(t,s)\Delta s}\right)$$

is contained in the interval $(\frac{m'_1}{\min f_\infty}, \frac{M'_1}{\max f_0})$ in Theorem 3. Theorem 3.2 in [4], the interval

$$\left(\frac{1}{M(\min f_0) \max\limits_{\xi \leqslant t \leqslant \omega} \int_\xi^\omega G(t,s)\Delta s}, \frac{1}{\max f_\infty \int_0^1 G(t,s)\Delta s}\right)$$

is contained in the interval $(\frac{m'_1}{\min f_0}, \frac{M'_1}{\max f_\infty})$ in Theorem 4. And also in our condition the function g may vanish identically on $[0,c]$ and $[d,\sigma^2(1)]$. So our results generalize the main results in [4,5].

Remark 6 If $T = R, \lambda = 1$, the results in [15] is one of the special case in this paper. Another, as $T = R, \beta = \delta = 0$, the main results in this paper can reduce to the results in [9,14]. Hence our paper extend some recent results.

References

[1] Agarwal R P,Bohner M. Basic calculus on time scales and some of its applications[J]. Results Math,1999,35:3-22.

[2] Amann H,Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces[J]. SIAM Rev,1976,18:620-709.
[3] Bohner M,Peterson A. Dynamic Equations on Times Scales, An Introduction with Applications[M]. Boston:Birkhauser,2001.
[4] Chen H,Chen Y. Positive solutions for eigenvalue problems on a measure chain[J]. Nonlinear Anal,2002,51:499-507.
[5] Chyan C J, Handerson J. Eigenvalue problems for nonlinear differential equations on a measure chain[J]. Math Anal Appl,2000,245:547-559.
[6] Kelevedjiev P. Nonnegative solutions to some singular second-order boundary value problems[J]. Nonlinear Anal,1999,36:481-494.
[7] Erbe L H,Peterson A. Positive solutions for a nonlinear differential euqation on a measure chain[J]. Math Comput Model,2000,32(5):571-585.
[8] Erbe L H,Peterson A. Green's functions and comparison theorems for differential equations on measure chains[J]. Dynam Contin, Discrete Impulse Systems,1999,6:121-138.
[9] Figueiredo D G,Lions P L,R. D. Nussbaun, A prior estimates and existence of positive solutions of semilinear elliptic equations[J]. J Math Pure Appl,1982,61:41-63.
[10] Henderson J,Wang H. Positive solutions of nonlinear eigenvalues problems[J]. J Math Anal Appl,1997,208:252-259.
[11] Keller H B. Some Positive Problems Suggested by Nonlinear Heat Generation[M]. NewYork:Benjiamin Elrnsfors,1969.
[12] Krasnoselskii M A. Positive Solutions of Operator Equations [M]. Noordhoff: Groningen, 1964.
[13] Lakshmikantham V S. Sivasundaram,Kaymakcalan B. Dynamic Systems on a Measure Chain[M]. Boston:Kluwer Academic Publishers, 1996.
[14] Mao A M,Luan S X. Positive solutions of nonlinear eigenvalue problems[J]. Math Anal(inChinese),2003,24(A2):167-174.
[15] Sun Y, Xu B L, Liu L S. Positive solutions of singular boundary value problems for Strum-Liouville equations[J]. J Sys Sci & Math Scis (in Chinese),2005,25(1):69-77.

测度链特征值问题的正解

王增桂

(聊城大学 数学科学学院,山东 聊城 252059)

摘 要 本文利用不动点指数理论研究了测度链特征值问题的正解存在性.本文结果推广了一些已有结论.

关键词 正解;特征值问题;不动点指数;测度链

中图分类号 O175.8 **文献标识码** A