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Singular Solutions of the (2+1)-Dimensional Euler Equation①

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Abstract　By applying the generalized CK direct method, we obtain the equivalence transformations and the relationships of new and old solutions of the (2+1)-dimensional Euler equation. Based on the solutions of Laplace equation, we get the formula to construct some explicit solutions and list some singular solutions of the Euler equation. By applying the equivalence transformations combined with the formula, we get some singular solutions with time varying of the (2+1)-dimensional Euler equation.

Key words　Euler equation; singular solution; equivalence transformation; Bäcklund transformation

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1　Introduction

It is well known that the Euler equation is one of extremely important models in fluid dynamics and is applied to many other physical fields such as atmospheric dynamics, statistical physics, crystal liquid and so on. Seeking explicit solutions and analyzing the corresponding properties of the Euler equation has been a major concern for a long time because the explicit solution plays an important role in the study of physical phenomena. It is very difficult to find its explicit solutions, but people continue to explore the method for finding explicit solutions of this problem. A symmetry group and Hamilton form of (3+1)-dimensional or (2+1)-dimensional incompressible Euler equation were given[1]. The weak Lax pairs was obtained by Y G Li[2]. In[3], a symmetry group theorem of the (2+1)-dimensional Euler equation is obtained via the Lax pair method which is thus utilized to find exact analytical vortex and circumfluence solutions. A weak Darboux transformation theorem of the (2+1)-dimensional Euler equation can be obtained for an arbitrary spectral parameter from the general symmetry group theorem. The Backlund transformation of the (2+1)-dimensional Euler equation is also presented by Lou etal in [4]. In 2006, Yurov and Yurova applied the so-called dressing method to construct exact solutions of equations of two-dimensional hydrodynamics of an incompressible fluid. The problem reduces to consecutively solving three linear partial differential equations[6].

In this paper we will consider the following (2+1)-dimensional Euler equation[3,4]

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$$w_t + [\psi, w] = 0 \,, \tag{1}$$

$$w = \Delta\psi \,, \tag{2}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator, and $[\psi, w] = \psi_x w_y - w_x \psi_y$.

In order to solve Eqs. (1) and (2) by applying the generalized CK direct method[6-8], we will discuss the relationships between new exact solutions and old ones. Following[4], we also construct the formula of the solutions Eqs. (1) and (2). In addition, a lot of singular solutions[5] and analytical solutions can be obtained.

2 Equivalence transformations of the Euler equation

Now we look for the equivalence transformations of Eqs. (1) and (2). Substituting Eq. (2) into Eq. (1), one can get the equivalence form of Eq. (1) as follows

$$\psi_{txx} + \psi_{tyy} + \psi_x\psi_{xxy} + \psi_x\psi_{yyy} - \psi_y\psi_{xxx} - \psi_y\psi_{xyy} = 0 \,. \tag{3}$$

Suppose that Eq. (3) has the following form solution

$$\psi(x,y,t) = \alpha(x,y,t) + \beta(x,y,t)\Psi(X(x,y,t), Y(x,y,t), T(x,y,t)) \,, \tag{4}$$

where the function $\Psi(X,Y,T)$ satisfies the following equation as Eq. (2)

$$\Psi_{TXX} + \Psi_{TYY} + \Psi_X\Psi_{XXY} + \Psi_X\Psi_{YYY} - \Psi_Y\Psi_{XXX} - \Psi_Y\Psi_{XYY} = 0, \tag{5}$$

and α, β, X, Y , T are the determined functions of x, y and t . Substituting Eq. (4) into Eq. (3) and letting the coefficients of Ψ and its derivatives be zero, we arrive at some equations to be determined. By solving these equations, the final results read

$$\alpha(x,y,t) = \frac{yF_{1t}(t) - xF_{2t}(t)}{C_1} + F_3(t) \,, \quad \beta(x,y,t) = \frac{T_t(t)}{C_1^2},$$

$$T(x,y,t) = T(t), X(x,y,t) = C_1 x + F_1(t), Y(x,y,t) = C_1 y + F_2(t) \,. \tag{6}$$

where F_1, F_2, F_3 and T are arbitrary functions of t , C_1 is a non-zero constant. With the help of Eq. (4), we get new exact solutions for the Euler equation (3) as follows

$$\psi(x,y,t) = \frac{yF_{1t}(t) - xF_{2t}(t)}{C_1} + F_3(t) + \frac{T_t(t)}{C_1^2}\Psi(X,Y,T) \,, \tag{7}$$

where the functions X, Y and T are determined by Eq. (6).

In summary, we have the following theorem of the equivalence transformations.

Theorem 1 If $\Psi(x,y,t)$ is a solution of the Euler equation (3), then so is ψ expressed by Eq. (7) with Eq. (6).

It can be seen from Theorem 1 that Eq. (7) can be viewed as an auto-Bäcklund transformation, which gives out the relationship between the new explicit solutions and old ones of the Euler equation (3). To see the relation between the equivalence transformations expressed by Theorem 1 and the Lie point symmetry group obtained by the standard Lie group approach, set

$$C_1 = 1 + \varepsilon c_1, T(t) = t + \varepsilon f(t), F_1(t) = \varepsilon f_1(t), F_2(t) = \varepsilon f_2(t), F_3(t) = \varepsilon f_3(t) \,, \tag{8}$$

where ε is an infinitesimal parameter, f, f_1, f_2 and f_3 are arbitrary functions of t . Then the symmetry of Eq. (7) reads as

$$\sigma(\Psi) = (c_1 x + f_1(t))\Psi_x + (c_1 y + f_2(t))\Psi_y + f(t)\Psi_t + (f_t - 2c_1)\Psi + f_{1t}(t)y - f_{2t}(t)x + f_3(t) \,. \tag{9}$$

By using the symmetry (9), one can look for the group invariant solutions w (or ψ) of Eq. (1). Here we omit the discussion in detail.

3 Singular solutions of the Euler equation

In this section, we will construct a kind of special solutions of Eqs. (1) and (2) and get the solving formula . Suppose that $w = w(x,y)$ is independent of t , one can get from Eq. (2) , that ψ must be have the following form

$$\psi = a(t) + b(t)x + c(t)y + F(x,y) \ , \tag{10}$$

where $a(t), b(t), c(t)$ and $F(x,y)$ are arbitrary functions satisfying $w = \Delta\psi = \Delta F(x,y)$. Substituting Eq. (10) into Eq. (1), we get

$$\frac{\partial(\Delta F)}{\partial t} + [a(t) + b(t)x + c(t)y + F, \Delta F] = 0 \ , \tag{11}$$

i. e.

$$b(t)\frac{\partial(\Delta F)}{\partial y} - c(t)\frac{\partial(\Delta F)}{\partial x} + [F, \Delta F] = 0 \ . \tag{12}$$

Since $b(t), c(t)$ are arbitrary functions, one can suppose

$$\Delta F = h \ , \tag{13}$$

where h is an arbitrary constant. Obviously, $\Delta F = h$ is a solution of Eq. (12) . One can take the transformation $F = S + (\alpha x^2 + \beta y^2 + \gamma xy)$, so that Eq. (13) can be changed into the homogenous harmonic equation $\Delta S = 0$, where $\alpha + \beta = \frac{h}{2}$.

As above, it shows that the following theorem of special solution holds.

Theorem 2 If $w(x,y)$ is an arbitrary function, then $\psi = f(x,y,t) + \sum_i k_i S_i(x,y)$ is also the solution of Eq. (1), where $f(x,y,t) = a(t) + b(t)x + c(t)y + \alpha x^2 + (\frac{h}{2} - \alpha)y^2 + \gamma xy$, k_i are arbitrary constants and $S_i(x,y)$ are arbitrary harmonic functions.

From Theorem 2, it can be seen that every harmonic function of x and y is always a seed solution, which can be applied to construct new solutions of Eq. (1) by Theorem 1. In this paper, we will list some singular solutions.

Case 1 Taking the function $S_1 = \ln\sqrt{(x-x_0)^2 + (y-y_0)^2}$, thus $F = \ln\sqrt{(x-x_0)^2 + (y-y_0)^2} + \alpha x^2 + \beta y^2 + \gamma xy$, where x_0, y_0 are real constants. Fig. 1 is the plot of the function S_1 when $x_0 = 1, y_0 = 1$. It shows that S_1 is the function with singularity, so is the corresponding solution of Eq. (5) (or Eq. (1)).

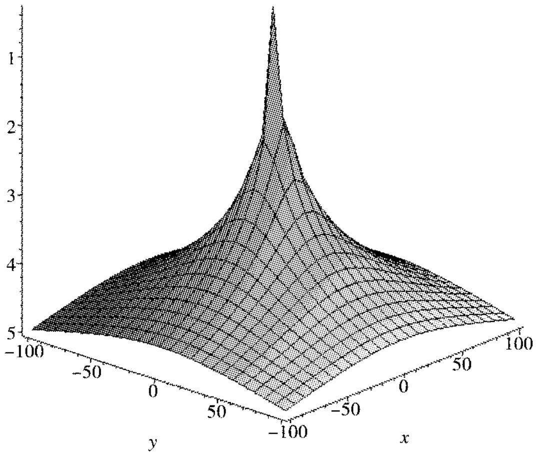

Fig. 1 **Plot of the solution** S_1

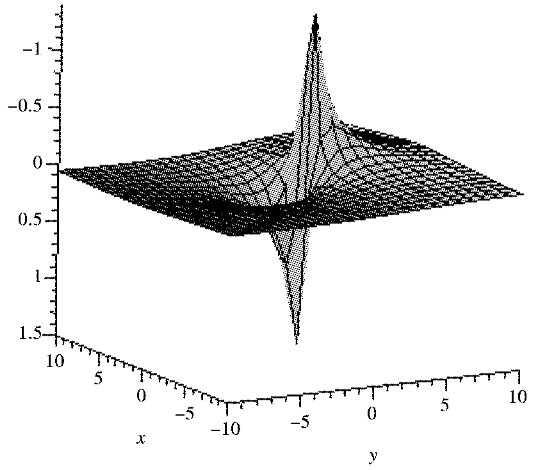

Fig. 2 **Plot of the solution** S_2'

Case 2 Letting $S_2 = \arg(z - z_0) = \arctan\frac{y - y_0}{x - x_0}$, $z_0 = x_0 + iy_0$. Although S_2 is a continuous function, its derivative S_2' with respect to x has the singularity. Fig. 2 is the plot of S_2' when $x_0 = 2, y_0 = 2$.

Case 3 Taking $S_3 = \dfrac{\sinh x}{\cosh x + \cos y}$,it is easy to see that S_3 and its derivative S_3' with respect to x have singularity, where $S_3' = \dfrac{\cosh(x)}{\cosh(x) + \cos y} - \dfrac{\sinh(x)^2}{(\cosh(x) + \cos y)^2}$. Fig. 3 and Fig. 4 are the plots of S_3 and S_3' respectively.

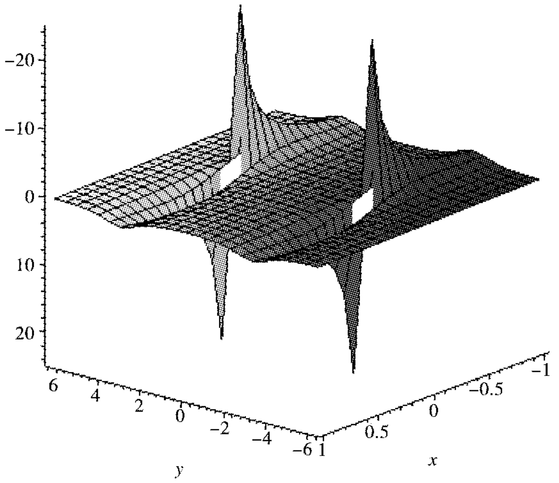

Fig. 3 **Plot of the solution** S_3

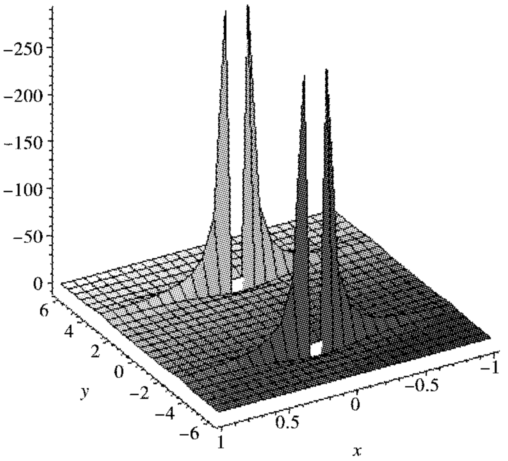

Fig. 4 **Plot of the solution** S_3'

By setting the different harmonic functions with singularity, one can obtain a lot of solutions to Eq. (1) Now we get out the plot of the solution depending of t . By the corresponding solutions of Eq. (1)

$$\psi = a(t) + b(t)x + c(t)y + \alpha x^2 + (\frac{h}{2} - \alpha)y^2 + \gamma xy + k_1 \ln \sqrt{(x - x_0)^2 + (y - y_0)^2} \\ + k_2 \arctan \frac{y - y_0}{x - x_0} + k_3 \frac{\sinh x}{\cosh x + \cos y}, \tag{14}$$

applying Theorem 1, yields the new singular solution as follows

$$\psi = \frac{yF_{1t}(t) - xF_{2t}(t)}{C_1} + F_3(t) + \frac{T_t(t)}{C_1^2}(a(T) + b(T)X + c(T)Y + \alpha X^2 + (\frac{h}{2} - \alpha)Y^2 + \gamma XY \\ + k_1 \ln \sqrt{(X - x_0)^2 + (Y - y_0)^2} + k_2 \arctan \frac{Y - y_0}{X - x_0} + k_3 \frac{\sinh X}{\cosh X + \cos Y}), \tag{15}$$

where $T(x,y,t) = T(t), X(x,y,t) = C_1 x + F_1(t), Y(x,y,t) = C_1 y + F_2(t)$.

For simplicity, setting$C_1 = \alpha = \gamma = h = 1$, $x_0 = y_0 = 0$, $F_1 = F_2 = F_3 = T = t$, $a(t) = b(t) = c(t) = t$and $k_i = 1$ in (15), one can get the following solution of Eq. (1)

$$\psi = y - x + 2t + 4tx + ty + xy + x^2 - \frac{1}{2}y^2 + \frac{7}{2}t^2 + \ln \sqrt{(x+t)^2 + (y+t)^2} + \\ + \arctan \frac{y+t}{x+t} + \frac{\sinh(x+t)}{\cosh(x+t) + \cos(y+t)}. \tag{16}$$

$$\psi' = -1 + 4t + 2x + y + \frac{(x+t)}{(x+t)^2 + (y+t)^2} - \frac{y+t}{(x+t)^2 + (y+t)^2} \\ + \frac{\cosh(x+t)}{\cosh(x+t) + \cos(y+t)} - \frac{\sinh(x+t)^2}{(\cosh(x+t) + \cos(y+t))^2}. \tag{17}$$

Fig. 5, Fig. 6 are the plots of (16) and Fig. 7, Fig. 8 are the plots of its derivative ψ' with respect to x when $t = 0$, $t = 1$, respectively.

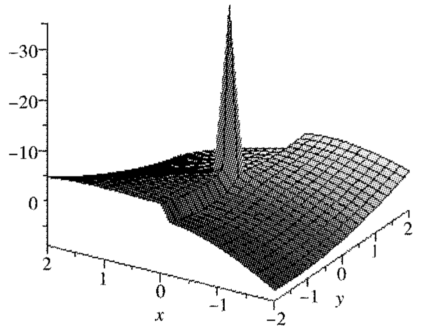

Fig. 5 Plot of ψ with $t=0$

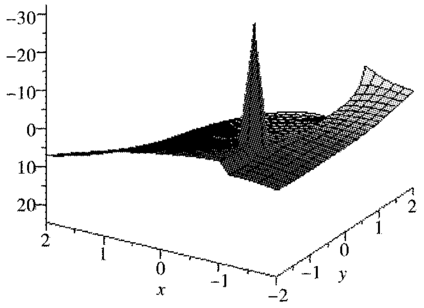

Fig. 6 Plot of ψ with $t=1$

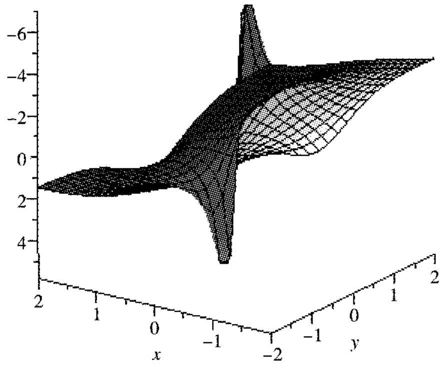

Fig. 7 Plot of ψ' with $t=0$

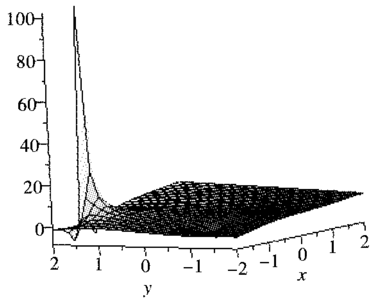

Fig. 8 Plot of ψ' with $t=1$

It can be seen from the plots that the singular points of the solution (16)and its derivative (17) vary as time goes on.

In addition, If one takes the function$S_4 = e^{-|x|}\cos y$, then the corresponding peaked solution of Eq. (1) can be obtained by applying the above procedure.

4 Conclusions

By applying the generalized CK direct method and construct a kind of special solution method to obtain many exact solutions of the (2+1)-dimensional Euler equation, which are new singular solutions. The listed solutions have singularity which may be important to explain some significant phenomena in physics. Although we discussed the case of singular solutions, it can be seen that a lot of analytic solutions are also found by taking the function S .

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四元数 Cholesky 分解的实保结构算法的再研究

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摘 要 文献[6]中,作者提出了四元数 Cholesky 分解的一种实保结构算法.本文对四元数 Cholesky 分解的实保结构算法进行了细致的研究,给出了基于高效运算的四元数 Hermitian 正定矩阵的 LDL^H 及 LL^H 分解的实保结构算法.我们将这两种实保结构算法的运算时间及精度与文献[6]中的算法及 Matlab 中的四元数工具包 QTFM 进行了比较.数值例子表明本文所提出的算法相对于利用低效运算[6]的算法及利用四元数代数运算的 QTFM 更加有效.

关键词 四元数矩阵;LDL^H; Cholesky 分解;实表示; 实保结构算法

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2+1 维欧拉方程的奇异解

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摘 要 借助于推广的 CK 方法,获得了 2+1 维欧拉方程的等价变换和新旧解之间的关系。基于拉普拉斯方程的解,给出了构造欧拉方程某些显式解的公式,并列出了部分奇异解.利用所求出的等价变换及其求解公式,得到了 2+1 维欧拉方程一些随时间演化的新奇异解.

关键词 欧拉方程; 奇异解; 等价变换; 贝克隆变换

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